Basic Ideas of Financial Mathematics

1 Percentage

The word “percent” simply means “out of 100”. Thus if you have 55% in a test, it means you obtained 55 marks out of a possible 100. This means you obtained

\[
\frac{55}{100} \text{ th's}
\]

of the marks available. So if the test is actually marked out of 40, then you have

\[
\frac{55}{100} \text{ of 40} = \frac{55}{100} \times 40 = 22 \text{ marks.}
\]

Thus, if we have R58.00, then 8% of that amount is

\[
R 58 \times \frac{8}{100} = R \ 4.64
\]

Similarly, 120% of R 300 is

\[
R 300 \times \frac{120}{100} = R \ 360.
\]

2 Compound Interest

Suppose you invest R400 in a bank. The bank then pays you interest on your investment. Let us suppose the rate of interest is 8% per annum. (“per annum” is just another way of saying “for each year”..) This means you earn interest of

\[
R 400 \times \frac{8}{100} = R \ 32.00
\]

The total amount you now have is

\[
R 400 + R \ 32 = R \ 432.
\]

It is important to understand that the bank pays you 8% of whatever you have in your account.

So, if you leave the money in the bank for another year, the bank now calculates the interest on R432. You then earn interest for the second year of

\[
R 432 \times \frac{8}{100} = R \ 34.56.
\]

Your account now holds the amount you had at the start of the second year plus the new interest earned, that is

\[
R 432.00 + R \ 34.56 = R \ 466.56.
\]

This process is known as calculating compound interest.

More generally, suppose we invest a sum \( P \), known as the principal at an interest rate \( i \) per annum. In the first year, the interest is \( I_1 = Pi \), so the amount is

\[
S_1 = P + I_1 = P + Pi = P(1 + i).
\]

We now calculate the interest for the second year on the new amount \( S_1 \). This is

\[
I_2 = S_1i = P(1 + i)i = Pi + Pi^2
\]
and the new amount is

\[ S_2 = S_1 + I_2 = P(1 + i) + P(1 + i)i = P(1 + i)(1 + i) = P(1 + i)^2. \]

Continuing in this way, the amount after \( n \) years is

\[ S_n = P(1 + i)^n. \]

Example 2.1

Find the amount if R800 is invested for 7 years at 6% per annum.

Solution

Here \( P = 800 \), \( n = 7 \) and \( i = \frac{6}{100} = 0.06 \). Thus

\[ S_7 = 800(1 + 0.06)^7 = 1202.90 \]

(where the calculation has been rounded to the nearest cent).

Example 2.2

If R200 is invested at 8% per annum, how long will it take for the amount to become worth R350?

Solution

Here \( P = 200 \), \( i = 0.08 \) and we wish the amount to be R350. Thus

\[ 350 = 200(1.08)^n. \]

It follows that

\[ 1.08^n = \frac{350}{200} = 1.75. \]

Taking logs (to any base) gives

\[ \log 1.75 = \log(1.08)^n = n \log 1.08 \]

using elementary properties of logs. Hence

\[ n = \frac{\log 1.75}{\log 1.08} = \frac{0.243308}{0.033424} \approx 7.27 \text{ years}. \]

The figures given are logs to the base 10, but the base is actually irrelevant. Note that no rounding should be done until the calculation has been completed.

3 Effective Interest Rate

Now suppose you were offered either an interest rate of 10% compounded once per year or an interest rate of 5% compounded every 6 months. Which would be the better? Or does it make no difference? Let us see.
Suppose then we have R1000. If the interest is compounded annually at 10%, then it is easy to see that the amount after 1 year is R1100.

However, if we compound twice a year at 5% we have

\(1000(1 + 0.05)^2 = 1102.50\)

which is more.

More generally, if the bank offers an interest rate of \(i\) compounded \(k\) times per year, then the amount accrued after one year on a principal \(P\) is

\[P\left(1 + \frac{i}{k}\right)^k.\]

**Example 3.1**

Suppose R1000 is invested at 6% per annum, compounded monthly. Find the amount after 1 year.

**Solution**

Since there are 12 months in a year, we obtain

\[1000\left(1 + \frac{0.06}{12}\right)^{12} = R1061.68.\]

In the previous example, we see that the interest rate is effectively 6.168%. We say that the **effective interest rate** is 6.168%, while the original interest rate of 6% is known as the **nominal interest rate**.

**Example 3.2**

Suppose a bank offers a nominal interest rate of 9%, which is compounded quarterly.

(a) What is the effective interest rate?

(b) If R500 is invested, what will the amount be after 5 years?

**Solution**

(a) To obtain the effective interest rate, it is easiest to see what happens to an investment of R100. Since there are 4 quarters in a year, the amount is

\[100\left(1 + \frac{0.09}{4}\right)^4 = 109.31,\]

so the effective interest rate is 9.31%.

(b) Over a period of 5 years there are 20 quarters, so the amount is

\[500\left(1 + \frac{0.09}{4}\right)^{20} \approx 780.25.\]

We can ask the question the other way around, that is, if we are given the effective interest rate, can we find the nominal rate?

**Example 3.3**
Suppose a principal $P$ is invested at an effective rate of 7% compounded 6 times per year. What is the nominal interest rate?

Solution

Let the nominal interest rate be $x$. We know the amount after 1 year is $P(1 + 0.07)$. So

$$P(1 + 0.07) = P\left(1 + \frac{x}{6}\right)^6.$$  

This gives $1 + \frac{x}{6} = 1.07^{1/6}$ and so $x = 0.6804$, that is a nominal interest rate of 6.804%.

4 Continuous Interest

Now suppose we start with an investment of R100 at a nominal rate of 10%. Let us calculate the interest rate if interest is compounded

(a) monthly  (b) weekly  (c) daily  (d) hourly  (e) every minute

(a) There are 12 months in a year, so we obtain

$$100\left(1 + \frac{0.10}{12}\right)^{12} \approx 110.47.$$ 

The effective interest rate is thus 10.47%.

(b) There are 52 weeks in a year, so now we have

$$100\left(1 + \frac{0.10}{52}\right)^{52} = 110.5065.$$ 

The effective interest rate is thus 10.5065%.

(c) Since there are 365 days in a year, we have

$$100\left(1 + \frac{0.10}{365}\right)^{365} = 110.5156$$ 

with an effective interest rate of 10.5156%.

(d) There are 24 hours in a day, so there are $24 \times 365 = 8760$ hours in a year. The amount is then

$$100\left(1 + \frac{0.10}{8760}\right)^{8760} = 110.5170$$ 

and the effective interest rate is 10.5170%.

(e) Since there are 60 minutes in each hour there are $60 \times 8760 = 525600$ minutes in a year. The same computation gives an effective interest of 10.5171%

Now consider the effective interest rates we obtained:

$$10.47, \quad 10.5065, \quad 10.5156, \quad 10.5170, \quad 10.5171$$

Clearly the numbers are increasing as the number of periods increases, but they are getting bigger more and more slowly.

So suppose we have $n$ periods, where $n$ is any very large number. Then the amount is

$$100\left(1 + \frac{0.10}{n}\right)^n.$$
If we now let \( n \to \infty \), we obtain
\[
\lim_{n \to \infty} 100 \left(1 + \frac{0.1}{n}\right)^n.
\]
We now use the fact that
\[
\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x.
\]
Setting \( x = 0.10 \), the amount is
\[
100e^{0.10} = 110.5171
\]
The effective interest rate is then 10.5171. This is known as the continuous rate of interest.

In general, if the nominal rate of interest is \( i \), then the continuous rate of interest is \( e^i \). To put it another way, if a principal \( P \) is invested at nominal interest rate \( i \), then after 1 year the amount is
\[
S_1 = Pe^i.
\]
After 2 years it will be
\[
S_2 = S_1e^i = P(e^i)^2 = Pe^{2i}
\]
and after \( n \) years it is
\[
P e^{ni}.
\]

Example 4.1

R100 000 is invested for 3 years. If the nominal rate of interest is 6% what will the amount be if interest is compounded
(a) annually  (b) quarterly  (c) continuously

Solution
(a) In this case we have
\[
100000(1 + 0.06)^3 = 119\,101.60.
\]
(b) Compounding quarterly we have 12 periods, and so the amount is
\[
100000 \left(1 + \frac{0.06}{4}\right)^{12} = 119\,561.80.
\]
(b) Continuous compounding gives
\[
100000(e^{0.06})^3 = 100000e^{0.18} = 119\,721.70
\]

5 Future and Present Values

If we invest a principal \( P \) at an interest rate \( i \) over \( n \) periods, then, as we have seen, the amount is
\[
S = P(1 + i)^n.
\]
This is the amount the investment will be worth in \( n \) periods of time, and is therefore also known as the future value of \( P \). We emphasize

The future value of an investment \( P \) is the amount it will be worth after \( n \) time periods taking into account interest earned

Now suppose, instead we wish to find how much money \( PV \), we must have in the bank now in order for our investment, at interest rate \( i \), to be worth \( P \) after \( n \) time periods. We must have
\[
PV(1 + i)^n = P.
\]
Thus
\[ PV = \frac{P}{(1 + i)^n} = P(1 + i)^{-n}. \]

This is known as the **present value** or **discounted value** of \( P \).

One can think of it this way. Suppose you have to make a payment \( P \), which must be made \( n \) time in the future. The present value of \( P \) is the money you need to deposit in the bank **now** in order to have enough money to pay the amount \( P \) after \( n \) time periods, taking into account interest earned. Again we emphasize

The present value of \( P \) due in \( n \) time periods is the amount that needs to be deposited **now** in order to amount to \( P \) after \( n \) time periods

**Example 5.1**

Find the present value of R1000 due in 5 years time at an interest rate of 8% per annum

**Solution**

Let the present value be \( PV \). Then the amount at 8% over 5 years is

\[ PV(1 + 0.08)^5 \]

We want this to be equal to R1000, so

\[ PV(1 + 0.08)^5 = 1000, \]

that is

\[ PV = \frac{1000}{(1.08)^5} = 1000 \times 1.08^{-5} = R680.58 \]

**Example 5.2**

Mr Mkhize wishes to ensure that there will be enough money in the bank in 18 years time to send his new-born daughter to UKZN. Taking into account inflation, he estimates that the fees then will be R1 000 000. Assuming an interest rate of 10%, how much must he deposit in the bank now?

**Solution**

Let the amount he must deposit be \( PV \). Then

\[ PV(1 + 0.10)^{18} = 1000000, \]

and so

\[ PV = \frac{1000000}{1.10^{18}} = 1000000 \times 1.1^{-18} = R179 858.80 \]

Note that this is far less than the one million he pays at the end. However, he may not have that much money right now. One way around his problem is to make a regular sequence of smaller payments into a special account, so that the payments plus their interest will eventually amount to the million he needs. We shall see how this operates in a little while.
6 Geometric Series

A series of the form
\[ S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \]  
(1)
is known as a **geometric series**. Note that each term is \( r \) times the previous one. Hence \( r \) is known as the common ratio. The **number of terms** is \( n \). Note that the right-hand side of (1) has \( n \) terms (not \( n - 1 \)), since we must also count the first term, which does not contain \( r \). Thus
\[ a + ar + ar^2 + ar^3 \]
has 4 terms, not 3. Count them!

Our main concern here is to find a formula for the sum of a geometric series. Let then
\[ S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}. \]  
(2)

Multiplying by \( r \) gives
\[ rS_n = ar + ar^2 + ar^3 + \cdots + ar^n-1 + ar^n. \]  
(3)

On subtracting (2) from (3), we have
\[ rS_n - S_n = ar^n - ar, \]
that is
\[ S_n(r - 1) = a(r^n - 1) \]
and so, provided \( r \neq 1 \),
\[ S_n = \frac{a(r^n - 1)}{(r - 1)} = \frac{a(1 - r^n)}{(1 - r)}, \quad r \neq 1 \]  
(4)

7 Annuities

Fundamentally an **annuity** is a sequence of equal payments made over equally spaced time intervals. Working with annuities is just like working with future and present values. The only difference is that we make a succession of deposits or a succession of payments.

Corresponding to future value, we consider how much money will be accrued, including interest, if we make regular deposits into a bank.

Corresponding to present value, we ask how much money we need to have in the bank now, taking into account interest, in order to make a sequence of regular payments in the future.

We consider two basic examples.

**Example 7.1**

Suppose I invest R1000 at the end of every year for 5 years. If the interest rate is 8%, how much will my investment be worth after 5 years?

**Solution**

In order to see what is happening, it is a good idea to draw a simple diagram.
Figure 1 shows the payments and when they are deposited.

Now consider the first payment. It remains in the bank for 4 years. The accumulated amount (i.e. its future value) is therefore

$$1000 \times (1 + 0.08)^4.$$  

The next deposit is in the bank for 3 years and is worth

$$1000 \times (1.08)^3.$$  

Continuing in this way, the other 3 deposits are worth

$$1000 \times (1.08)^2, \quad 1000 \times (1.08)^1 \quad \text{and} \quad 1000.$$  

Notice that the last payment accrues no interest, as it is made at the end of the fifth year. The total is therefore

$$1000 \times (1.08)^4 + 1000 \times (1.08)^3 + 1000 \times (1.08)^2 + 1000 \times (1.08)^1 + 1000$$

Writing this in reverse order gives an amount of

$$1000 + 1000 \times (1.08) + 1000 \times (1.08)^2 + 1000 \times (1.08)^3 + 1000 \times (1.08)^4.$$  

This is a geometric series. So using (4), where \(a = 1000, r = 1.08\) and \(n = 5\) (since there are 5 terms in the series), the amount is

$$S = \frac{1000(1.08^5 - 1)}{1.08 - 1} = \frac{1000(1.08^5 - 1)}{0.08}. \quad (5)$$

A calculation shows that

$$S = R\ 5866.60$$

Now let us introduce some terminology.

- The **payment interval** is the time between payments (in the case above, 1 year),
- The **term** is the length of time from the beginning to the end (in the case above, 5 years),
- The **annual rent** is the amount paid per year (in the case above R 1000),
- An annuity is an **ordinary annuity** if payments are made at the **end** of each time period, as is the case above. If they are made at the **beginning**, it is a **due annuity**.
- The **amount** is the total of all the payments made plus their interest, that is the future value of all deposits.
Now let us reconsider (5) and see if we can derive a general formula. Let the annual rent be \( P \). In the case above \( P = 1000 \). Let the term be \( n \). In the case above \( n = 5 \). Let the interest rate per time interval be \( i \). In the case above \( i = 0.08 \). Then using exactly the same argument, the amount is

\[
S = \frac{P((1 + i)^n - 1)}{i}.
\]

In the case that \( P = 1 \) we obtain

\[
S = \frac{(1 + i)^n - 1}{i}.
\]

Of course \( S \) depends on \( n \) and \( i \), so we write the amount as

\[
s_{n|i} = \frac{(1 + i)^n - 1}{i} \tag{6}
\]

For our second example consider the following situation.

**Example 7.2**

Mr Mkhize needs to make regular payments at the end of each month in order to pay for his furniture. How much money does he need to have in the bank now in order to pay for his furniture if the monthly payments are R 200 for 2 years at a nominal interest rate of 15% compounded monthly?

**Solution**

Again it is a good idea to draw a diagram.

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>\cdots</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>\cdots</td>
<td>200</td>
<td>200</td>
</tr>
</tbody>
</table>

Figure 2

Now we need to ensure that there is enough money in the bank now to be able to make the payments, taking into account the interest earned. Thus we need to find the sum of all the present values.

Note that the interest rate is \( \frac{0.15}{12} = 0.0125 \) (since it is compounded monthly) over 24 months. Let the money deposited initially be \( PV \). The first payment is R200. The money, \( PV_1 \) to make this earns interest for 1 month. So

\[
PV_1(1 + 0.0125) = 200
\]

that is

\[
PV_1 = \frac{200}{1.0125} = 200(1.0125)^{-1}.
\]

The second payment is again R200, but remains in Mr Mkhize’s account for 2 months. The money, \( PV_2 \) is then given by

\[
PV_2(1 + 0.0125)^2 = 200
\]

so

\[
PV_2 = \frac{200}{1.0125} = 200(1.0125)^{-2}.
\]
Continuing in this way, we arrive at the 24th payment, which earns interest for 24 months, given a total present value of

\[ PV = PV_1 + PV_2 + \cdots + PV_{24} = 200(1.0125)^{-1} + 200(1.0125)^{-2} + \cdots + 200(1.0125)^{-24} \]

Again we have a geometric series in which \( a = 1, \ r = (1.0125)^{-1} \) and \( n = 24 \). Using (4), the total present value is

\[ PV = 200(1.0125)^{-1} \left( \frac{1 - (1.0125)^{-24}}{1 - (1.0125)^{-1}} \right) = 200(1.0125)^{-1} \left( \frac{1 - (1.0125)^{-24}}{1 - (1.0125)^{-1}} \right). \]

To clean up this somewhat messy expression, multiply the numerator and denominator by 1.0125. We obtain

\[ PV = 200 \left( \frac{1 - (1.0125)^{-24}}{1.0125 - 1} \right) = 200 \left( \frac{1 - (1 + 0.0125)^{-24}}{0.0125} \right). \]

A calculation shows that

\[ PV = 4124.85. \]

We emphasize again, this is the quantity of money needed in the bank now in order to affect 24 payments in the future. It is simply the sum of all the present values of the payments, and is therefore called the present value of the annuity.

Again it is fairly easy to see what the general formula is for the present value of an annuity. Suppose the payments are \( p \) (in the example above \( p = 200 \), the interest rate is \( i \) (in the case above \( i = 0.0125 \)) and the number of payments is \( n \) (in the case above \( n = 24 \)). Then, using just the same argument,

\[ PV = p \times \frac{1 - (1 + i)^{-n}}{i}. \]

In the case that \( p = 1 \), \( PV \) depends on \( n \) and \( i \), so we write the present value as

\[ a_{n|i} = \frac{1 - (1 + i)^{-n}}{i}. \]

Example 7.3

What is the amount for an investment of R 5000 where payments are made quarterly over 5 years at a nominal interest rate of 6% compounded quarterly?

Solution

Here we are being asked to find the future value. We have 20 periods (since there are 5 years each with 4 quarters), and the interest rate is \( i = \frac{0.06}{4} = 0.015 \) since we are compounding quarterly. Thus

\[ n = 20 \quad \text{and} \quad i = \frac{0.06}{4} = 0.015 \]

So, by (6)

\[ S = 5000 \times a_{20|0.015} = 5000 \times \left( \frac{(1 + 0.015)^{20} - 1}{0.015} \right) = R \, 115 \, 618.34 \]
Example 7.4

Find the present value of an annuity, where payments of R 10 000 are to be paid at the end of every year for 7 years at an interest rate of 12.5%.

Solution

Now we are being asked to find the present value. Here $n = 7$ and $i = 0.125$, so by (8),

$$PV = 10000 \times a_{n|i} = 10000 \times \frac{(1 - (1 + 0.125)^7)}{0.125} = \text{R} 44923.01$$