1. The position an object moving in space is given by

\[ \mathbf{r} = (4t^3 - 8t^2)\mathbf{i} + (\sin t + t^2)\mathbf{j} + 12t^3\mathbf{k} \text{ (m)}. \]

(a) Find the vector velocity and the vector acceleration of the object as functions of time.  

**Solution**

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} = (12t^2 - 16t)\mathbf{i} + (\cos t + 2t)\mathbf{j} + 36t^2\mathbf{k} \text{ (m/s)}; \]

\[ \mathbf{a} = \frac{d\mathbf{v}}{dt} = (24t - 16)\mathbf{i} + (-\sin t + 2)\mathbf{j} + 72t\mathbf{k} \text{ (m/s}^2). \]

(b) What is the magnitude of its velocity when \( t = 5 \) s.

**Solution**

At time \( t = 5 \), \( \mathbf{v} = (12 \times 5^2 - 16 \times 5)\mathbf{i} + (\cos 5 + 2 \times 5)\mathbf{j} + 36 \times 5^2\mathbf{k} = 220\mathbf{i} + (\cos 5 + 10)\mathbf{j} + 900\mathbf{k}. \) The magnitude of \( \mathbf{v} \) is then

\[ |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = 926.55 \text{ m/s.} \]

2. An object is undergoing a curvilinear motion with the components of its acceleration given by \( a_x = 2t^2 - 3t^3, \ a_y = 15t^4 - 4t \). At time \( t = 0 \), the object is at the origin with its velocity given by \( \mathbf{v} = \mathbf{i} - 3\mathbf{j} \). Find the magnitude of the velocity and the position of the object when \( t = 2 \) s.

**Solution**

We have \( a_x = \frac{dv_x}{dt} \) or \( dv_x = (2t^2 - 3t^3)dt \) and integrating \( \int_0^{v_x} dv_x = \int_0^t (2t^2 - 3t^3)dt \) which gives

\[ v_x = 1 + \frac{2}{3}t^3 - \frac{3}{4}t^4. \]

Similarly \( a_y = \frac{dv_y}{dt} \) or \( dv_y = (15t^4 - 4t)dt \) and integrating \( \int_0^{v_y} dv_y = \int_0^t (15t^4 - 4t)dt \) which gives

\[ v_y = -3 + 3t^5 - 2t^2. \]

At time \( t = 2 \), \( v_x = 1 + \frac{2}{3}2^3 - \frac{3}{4}2^4 = -5.666, \ v_y = -3 + 3 \times 2^5 - 2 \times 2^2 = 85 \) and hence the magnitude of the velocity is |\( \mathbf{v} | = \sqrt{v_x^2 + v_y^2} = 85.18 \text{ m/s.} \]

To find the position, use \( dx = v_x dt \) or \( dx = (1 + \frac{2}{3}t^3 - \frac{3}{4}t^4)dt \) and integrating \( \int_0^x dx = \int_0^2 (1 - \)
\[ \frac{2}{3}t^3 - \frac{3}{4}t^4 \, dt \] which gives \( x = 2 + \frac{2}{12}t^2 - \frac{3}{20}t^5 = -0.133 \text{ m}. \) Also \( dy = v_y \, dt = (-3 + 3t^5 - 2t^2) \, dt \) and integrating \( \int_0^3 dy = \int_0^3 (-3 + 3t^5 - 2t^2) \, dt = -3 \times 2 + \frac{1}{7}t^6 - \frac{2}{2}t^3 = . \)

\[ x = -0.13 \text{ m} \quad y = 20.66 \text{ m}. \]

Note that we write the position of an object in cartesian coordinates as \( r = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) and NOT \( r = s_x \mathbf{i} + s_y \mathbf{j} + s_z \mathbf{k} \) NOR \( r = r_x \mathbf{i} + r_y \mathbf{j} + r_z \mathbf{k} \)

3. A steel ball is released from the initial velocity of 2 cm/s in a container of oil. Its downward acceleration is given in terms of its velocity \( v \) in cm/s as \( a = 3 - v \text{ cm/s}^2 \). What is the ball’s downward velocity 5 s after it is released? (3)

**Solution**

\[ a = \frac{dv}{dt} \therefore dv = (3 - v) \, dt \]

and integrating

\[ \int_2^5 (3 - v) \, dt = \int_0^5 dt \]

which gives \( [\ln|3 - v|]_2^5 = -5 \) or \( \ln|3 - v| - \ln1 = -5 \) or \( 3 - v = e^{-5} \) or \( v = 3 - e^{-5} = 2.993 \text{ m/s}. \)

4. A person standing on a vertical cliff at height \( h = 3 \text{ m} \) above the ground level jump high to the ground, leaving the cliff at speed 5 m/s and making and angle of 30° with the horizontal. He land on the ground \( T \) second later. Find \( T \), and the ground distance from the cliff to his landing spot. (6)

**Solution**

Choose a frame of reference with the origin on the ground just below the edge of the cliff, the \( x \)-axis horizontal on the ground and the \( y \)-axis vertical. The equations of motion reads

\[
\begin{align*}
x &= 5t \cos 30^\circ = \frac{5\sqrt{3}}{2}t \\
y &= 3 + 5t \sin 30^\circ - \frac{1}{2}gt^2 = 3 + \frac{5}{2}t - 4.905t^2
\end{align*}
\]

The person land on the ground at the positive time \( T \) when \( y = 0 \). Solve the equation \( 3 + \frac{5}{2}t - 4.905t^2 = 0 \) using the quadratic formula to find that \( t = \frac{\frac{5}{2} \pm \sqrt{\frac{5}{2}^2 + 4 \times 3 \times 4.905}}{-9.81} = -0.567 \) or 1.077. Discard the negative solution to have 

\( T = 1.07 \text{ s} \)

and substitute in the expression for \( x \) to get the distance from the cliff to the landing spot as

\( D = \frac{5\sqrt{3}}{2}(1.07) = 4.66 \text{ m}. \)

5. A skater initially at 5 rev/s (revolution per second) slows down with constant deceleration and reduces its angular velocity to 2 rev/s in 3 seconds. Find her angular deceleration \( \alpha \). How many revolutions does she take to reduce its angular velocity to 2 rev/s? (6)

Note the conversion 1rev/s = 2\pi rad/s. Also \( \alpha = \text{constant} \) and \( \alpha = \frac{d\omega}{dt} \) and integrating gives

\[ \int_{10\pi}^{4\pi} d\omega = \alpha \int_0^3 dt \] or \( 4\pi - 10\pi = 3\alpha \) which gives

\( \alpha = -2\pi \text{ rad/s}^2. \)
Now $\alpha = \frac{d\omega}{dt} = -2\pi$ and integrating, we get

$$\omega = 10\pi - 2\pi t.$$ Integrating again, we get

$$\int_0^{\Delta \theta} d\theta = \int_0^3 (10\pi - 2\pi t)dt = \left[10\pi t - \pi t^2\right]_0^3 = 30\pi - 9\pi = 21\pi = \frac{21}{2}(2\pi)$$ Hence

$$\Delta \theta = \frac{21}{2}(2\pi) \text{ rad},$$

which correspond to ten revolutions and a half.

6. State if each of the following statements is true (T) or false (F)

Solution

(a) The speed of a particle is a vector. (F)
(b) The acceleration of a particle is always tangent to its path. (F)
(c) The velocity of a point moving in space is always tangent to its path. (T)
(d) For object undergoing projectile motion in the $xy$–plane with the $y$ axis vertical, the highest point on the path is obtained when the $y$-component of the velocity vanishes. (T)
(e) The acceleration of a particle is the rate of change of its velocity. (T)
(f) The path of an object undergoing projectile motion is a straight line. (F)