**Problem 8.1** Use the method described in Active Example 8.1 to determine $I_y$ and $k_y$ for the rectangular area.

![Diagram of rectangular area]

**Solution:** The height of the vertical strip of width $dx$ is 0.6 m, so the area is $dA = (0.6 \, dx)$.

We can use this expression to determine $I_y$:

$$I_y = \int_{A} x^2 \, dA = \int_{0.2m}^{0.4m} x^2 \, dx = (0.6 \, m) \left[ \frac{x^3}{3} \right]_{0.2m}^{0.4m} = 0.0416 \, m^4$$

The radius of gyration about the $y$ axis is

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{0.0416 \, m^4}{(0.4 \, m)(0.6 \, m)}} = 0.416 \, m$$

$\boxed{I_y = 0.0416 \, m^4, k_y = 0.416 \, m}$

**Problem 8.2** Use the method described in Active Example 8.1 to determine $I_x$ and $k_x$ for the rectangular area.

![Diagram of rectangular area]

**Solution:** It was shown in Active Example 8.1 that the moment of inertia about the $x$ axis of a vertical strip of width $dx$ and height $f(x)$ is

$$(I_x)_{strip} = \frac{1}{3} \left[ f(x) \right]^3 \, dx.$$

For the rectangular strip, $f(x) = 0.6 \, m$. Integrating to determine $I_x$ for the rectangular area:

$$I_x = \int_{0.2m}^{0.4m} \frac{1}{3} (0.6 \, m)^3 \, dx = \frac{1}{3} (0.6 \, m)^3 \left[ x \right]_{0.2m}^{0.4m} = 0.0288 \, m^4$$

The radius of gyration about the $x$ axis is

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{0.0288 \, m^4}{(0.4 \, m)(0.6 \, m)}} = 0.346 \, m$$

$\boxed{I_x = 0.0288 \, m^4, k_x = 0.346 \, m}$

**Problem 8.3** In Active Example 8.1, suppose that the triangular area is reoriented as shown. Use integration to determine $I_y$ and $k_y$.

**Solution:** The height of a vertical strip of width $dx$ is $h - (h/b)x$, so the area

$$dA = \left( h - \frac{h}{b}x \right) \, dx.$$

We can use this expression to determine $I_y$:

$$I_y = \int_{A} x^2 \, dA = \int_{0}^{b} x^2 \left( h - \frac{h}{b}x \right) \, dx = h \left[ \frac{x^3}{3} - \frac{x^4}{4b} \right]_{0}^{b} = \frac{1}{12} h b^3$$

The radius of gyration about the $y$ axis is

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{\frac{1}{12} h b^3}{\frac{1}{2} b h}} = \frac{b}{\sqrt{6}}$$

$\boxed{I_y = \frac{1}{12} h b^3, k_y = \frac{b}{\sqrt{6}}}$
Problem 8.4  (a) Determine the moment of inertia $I_y$ of the beam’s rectangular cross section about the $y$ axis.

(b) Determine the moment of inertia $I_y$ of the beam’s cross section about the $y^\prime$ axis. Using your numerical values, show that $I_y = I_y + d_y^2 A$, where $A$ is the area of the cross section.

Problem 8.5  (a) Determine the polar moment of inertia $I_O$ of the beam’s rectangular cross section about the origin $O$.

(b) Determine the polar moment of inertia $I_O$ of the beam’s cross section about the origin $O'$. Using your numerical values, show that $I_O = I_O + (d_x^2 + d_y^2) A$, where $A$ is the area of the cross section.

Problem 8.6  Determine $I_y$ and $k_y$.

Solution:

(a) $I_y = \int_0^{40 \text{ mm}} \int_0^{60 \text{ mm}} x^2 dydx = 1.28 \times 10^6 \text{ mm}^4$

(b) $I_y = \int_{-20 \text{ mm}}^{0 \text{ mm}} \int_{-30 \text{ mm}}^{0 \text{ mm}} x^2 dydx = 3.2 \times 10^5 \text{ mm}^4$

(c) $I_y = I_y + d_y^2 A$

$1.28 \times 10^6 \text{ mm}^4$

$= 3.2 \times 10^5 \text{ mm}^4 + (20 \text{ mm})^2(40 \text{ mm})(60 \text{ mm})$

$= 1.28 \times 10^6 \text{ mm}^4$

$+ (20 \text{ mm})^2(30 \text{ mm})(60 \text{ mm})$

Solution:

(a) $J_O = \int_0^{40 \text{ mm}} \int_0^{60 \text{ mm}} (x^2 + y^2) dydx = 4.16 \times 10^6 \text{ mm}^4$

(b) $J_O = \int_{-20 \text{ mm}}^{0 \text{ mm}} \int_{-30 \text{ mm}}^{0 \text{ mm}} (x^2 + y^2) dydx = 1.04 \times 10^6 \text{ mm}^4$

(c) $J_O = J_O + (d_x^2 + d_y^2) A$

$4.16 \times 10^6 \text{ mm}^4 = 1.04 \times 10^6 \text{ mm}^4 + (20 \text{ mm})^2$

$+ (30 \text{ mm})^2(40 \text{ mm})(60 \text{ mm})$

$4.16 \times 10^6 \text{ mm}^4$

$= 1.04 \times 10^6 \text{ mm}^4 + (20 \text{ mm})^2$

$+ (30 \text{ mm})^2(40 \text{ mm})(60 \text{ mm})$

$= 1.04 \times 10^6 \text{ mm}^4$

$+ (20 \text{ mm})^2(30 \text{ mm})(60 \text{ mm})$

Solution:

$A = (0.3 \text{ m})(1 \text{ m}) + \frac{1}{2}(0.3 \text{ m})(1 \text{ m}) = 0.45 \text{ m}^2$

$I_y = \int_0^{1 \text{ m}} \int_0^{0.3 \text{ m} + 0.6 \text{ m}} x^2 dydx = 0.175 \text{ m}^4$

$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{0.175 \text{ m}^4}{0.45 \text{ m}^2}} = 0.624 \text{ m}$

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Problem 8.7 Determine $J_O$ and $k_O$.

Solution:

$$A = (0.3 \text{ m})(1 \text{ m}) + \frac{1}{2}(0.3 \text{ m})(1 \text{ m}) = 0.45 \text{ m}^2$$

$$J_O = \int_0^1 \int_0^{0.3 \text{ m} + 0.3x} (x^2 + y^2) \, dy \, dx = 0.209 \text{ m}^4$$

$$k_O = \sqrt{\frac{0.209 \text{ m}^4}{0.45 \text{ m}^2}} = 0.681 \text{ m}$$

Problem 8.8 Determine $I_{xy}$.

Solution:

$$I_{xy} = \int_0^1 \int_0^{0.3 \text{ m} + 0.3x} xy \, dy \, dx = 0.0638 \text{ m}^4$$

Problem 8.9 Determine $I_y$.

Solution: The height of a vertical strip of width $dx$ is $2 - x^2$, so the area is

$$dA = (2 - x^2) \, dx.$$  

We can use this expression to determine $I_y$:

$$I_y = \int x^2 \, dA = \int_0^1 x^2 (2 - x^2) \, dx = \left[ \frac{2}{3} \left( \frac{x^3}{3} \right) \right]_0^1$$

$$= 0.467.$$  

$I_y = 0.467$.

Problem 8.10 Determine $I_x$.

Solution: It was shown in Active Example 8.1 that the moment of inertia about the $x$ axis of a vertical strip of width $dx$ and height $f(x)$ is

$$\left( I_x \right)_{\text{strip}} = \frac{1}{4}[f(x)]^3 \, dx.$$  

In this problem $f(x) = 2 - x^2$. Integrating to determine $I_x$ for the area,

$$I_x = \int_0^1 \frac{1}{3} (2 - x^2)^3 \, dx$$

$$= \frac{1}{3} \int_0^1 (8 - 12x^2 + 6x^4 - x^6) \, dx$$

$$= \frac{1}{3} \left[ 8 \left( \frac{x^3}{3} \right) - 12 \left( \frac{x^4}{4} \right) + 6 \left( \frac{x^5}{5} \right) - \frac{x^7}{7} \right]_0^1 = 1.69.$$  

$I_x = 1.69$.  

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Problem 8.11  Determine $J_O$.

**Solution:** See the solutions to Problems 8.9 and 8.10. The polar moment of inertia is

$$J_O = I_x + I_y = 1.69 + 0.467 = 2.15.$$  

$J_O = 2.15$.

Problem 8.12  Determine $I_{xy}$.

**Solution:** It was shown in Active Example 8.1 that the product of inertia of a vertical strip of width $dx$ and height $f(x)$ is

$$(I_{xy})_{strip} = \frac{1}{2} (f(x))^2 dx.$$  

In this problem, $f(x) = 2 - x^2$. Integrating to determine $I_{xy}$ for the area,

$$I_{xy} = \int_{0}^{1} \frac{1}{2} (2 - x^2)^2 dx = \frac{1}{2} \int_{0}^{1} (4x - 4x^3 + x^5)dx$$  

$$= \frac{1}{2} \left[ \frac{4x^2}{2} - \frac{4x^4}{4} + \frac{x^6}{6} \right]_0^1 = 0.583.$$

$I_{xy} = 0.583$.

Problem 8.13  Determine $I_y$ and $k_y$.

**Solution:** First we need to locate the points where the curve intersects the $x$ axis.

$$\frac{1}{3}x^3 + 4x - 7 = 0 \Rightarrow x = \frac{-4 \pm \sqrt{16 - 4(1/4)(-75)}}{2(-1/4)} = 2.14$$  

Now $A = \int_{2}^{14} \int_{0}^{\sqrt[3]{4x+7}} dydx = 72$

$$I_y = \int_{2}^{14} \int_{0}^{\sqrt[3]{4x+7}} x^2 dydx = 5126$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{5126}{72}} = 8.44$$
Problem 8.14  Determine $I_x$ and $k_x$.

Solution:  See Solution to Problem 8.13

\[
I_x = \int_{14}^{2} \int_{0}^{4} \frac{x^2}{4} + x - 7 \, dy \, dx = 1333
\]

\[
k_x = \sqrt{\frac{I_x}{A}} = \frac{1333}{\frac{21}{2}} = 4.30
\]

Problem 8.15  Determine $J_O$ and $k_O$.

Solution:  See Solution to 8.13 and 8.14

\[
J_O = I_x + I_y = 1333 + 5126 = 6459
\]

\[
k_O = \sqrt{\frac{J_O}{A}} = \sqrt{\frac{6459}{\frac{21}{2}}} = 9.47
\]

Problem 8.16  Determine $I_{xy}$.

Solution:

\[
I_{xy} = \int_{14}^{2} \int_{0}^{4} \frac{x^2}{4} + x - 7 \, x \, dy \, dx = 2074
\]

Problem 8.17  Determine $I_y$ and $k_y$.

Solution:  First we need to locate the points where the curve intersects the line.

\[
-\frac{1}{4} x^2 + 4x - 7 = 5 \Rightarrow x = -4 \pm \sqrt{16 - 4(-1/4)(-12)} = 4.12
\]

\[
A = \int_{4}^{12} \int_{5}^{-x^2/4 + 4x - 7} ddy \, dx = 21.33
\]

\[
I_y = \int_{4}^{12} \int_{5}^{-x^2/4 + 4x - 7} x^2 \, ddy \, dx = 1434
\]

\[
k_y = \sqrt{\frac{I_y}{A}} = \frac{1434}{21.33} = 8.20
\]

Problem 8.18  Determine $I_x$ and $k_x$.

Solution:  See Solution to Problem 8.17

\[
I_x = \int_{4}^{12} \int_{5}^{-x^2/4 + 4x - 7} y^2 \, ddy \, dx = 953
\]

\[
k_x = \sqrt{\frac{I_x}{A}} = \frac{953}{21.33} = 6.68
\]
Problem 8.19  (a) Determine $I_y$ and $k_y$ by letting $dA$ be a vertical strip of width $dx$.
(b) The polar moment of inertia of a circular area with its center at the origin is $J_O = \frac{1}{2} \pi R^4$. Explain how you can use this information to confirm your answer to (a).

Solution: The equation of the circle is $x^2 + y^2 = R^2$, from which $y = \pm \sqrt{R^2 - x^2}$. The strip $dx$ wide and $y$ long has the elemental area $dA = 2x \sqrt{R^2 - x^2} dx$. The area of the semicircle is

$$A = \frac{\pi R^2}{2} = \int_A x^2 dA = 2 \int_0^R x^2 \sqrt{R^2 - x^2} dx$$

$$= 2 \left[ -\frac{x(R^2 - x^2)^{1/2}}{4} + \frac{R^2 x(R^2 - x^2)^{1/2}}{8} + \frac{R^4}{8} \sin^{-1}\left(\frac{R}{R}\right) \right]_0^R$$

$$= \frac{\pi R^4}{8}$$

$$k_y = \frac{\int x^2 A}{A} = \frac{R}{2}$$

(b) If the integration were done for a circular area with the center at the origin, the limits of integration for the variable $x$ would be from $-R$ to $R$, doubling the result. Hence, doubling the answer above,

$$I_y = \frac{\pi R^4}{8}$$

By symmetry, $I_x = I_y$, and the polar moment would be

$$J_O = 2I_y = \frac{\pi R^4}{4}$$

which is indeed the case. Also, since $k_x = k_y$ by symmetry for the full circular area,

$$k_O = \sqrt{\frac{I_x}{A} + \frac{I_y}{A}} = \sqrt{\frac{2I_x}{A}} = \sqrt{\frac{2J_O}{A}}$$

as required by the definition. Thus the result checks.

Problem 8.20  (a) Determine $I_x$ and $k_x$ for the area in Problem 8.19 by letting $dA$ be a horizontal strip of height $dy$.
(b) The polar moment of inertia of a circular area with its center at the origin is $J_O = \frac{1}{2} \pi R^4$. Explain how you can use this information to confirm your answer to (a).

Solution: Use the results of the solution to Problem 8.19, $A = \frac{\pi R^2}{2}$. The equation for the circle is $x^2 + y^2 = R^2$, from which $x = \pm \sqrt{R^2 - y^2}$. The horizontal strip is from 0 to $R$, hence the element of area is

$$dA = \sqrt{R^2 - y^2} dy.$$ 

$$I_x = \int_A x^2 dA = \int_{-R}^R y^2 \sqrt{R^2 - y^2} dy$$

$$= \left[ -\frac{y(R^2 - y^2)^{1/2}}{4} - \frac{R^2 y(R^2 - y^2)^{1/2}}{8} + \frac{R^4}{8} \sin^{-1}\left(\frac{y}{R}\right) \right]_{-R}^R$$

$$= \left[ \frac{R^4 \pi}{8} + \frac{R^4 \pi}{8} \right] = \frac{\pi R^4}{8}$$

$$k_x = \frac{\int x^2 A}{A} = \frac{R}{2}$$

(b) If the area were circular, the strip would be twice as long, and the moment of inertia would be doubled:

$$I_x = \frac{\pi R^4}{4}$$

By symmetry $I_x = I_y$.

and $J_O = 2I_x = \frac{\pi R^4}{2}$.

which is indeed the result. Since $k_x = k_y$ by symmetry for the full circular area, the

$$k_O = \sqrt{\frac{I_x}{A} + \frac{I_y}{A}} = \sqrt{\frac{2I_x}{A}} = \sqrt{\frac{2J_O}{A}}$$

as required by the definition. Thus the result checks.
Problem 8.21 Use the procedure described in Example 8.2 to determine the moment of inertia $I_x$ and $I_y$ for the annular ring.

Solution: We first determine the polar moment of inertia $J_O$ by integrating in terms of polar coordinates. Because of symmetry and the relation $J_O = I_x + I_y$, we know that $I_x$ and $I_y$ each equal $\frac{1}{2} J_O$. Integrating as in Example 8.2, the polar moment of inertia for the annular ring is

$$J_O = \int_A r^2 dA = \int_{R_i}^{R_o} r^2 (2\pi r) dr = \frac{1}{2} \pi (R_o^4 - R_i^4)$$

Therefore

$$I_x = I_y = \frac{1}{4} \pi (R_o^4 - R_i^4)$$
Problem 8.22  What are the values of $I_y$ and $k_y$ for the elliptical area of the airplane’s wing?

Solution:

$$I_y = \int_A x^2 \, dA = \int_0^b \int_x^x x^2 \, dy \, dx$$

$$I_y = 2 \int_0^b \int_x^x x^2 \, dy \, dx$$

$$I_y = 2 \int_0^b x^2 b \left(1 - \frac{x^2}{a^2}\right)^{1/2} \, dx$$

$$I_y = 2b \int_0^b x^2 \sqrt{1 - \frac{x^2}{a^2}} \, dx$$

Rewriting

$$I_y = \frac{2b}{a} \int_0^b x^2 \sqrt{a^2 - x^2} \, dx$$

$$I_y = \frac{2b}{a} \left[ \frac{x(a^2 - x^2)^{1/2}}{4} + \frac{a^2 \sqrt{a^2 - x^2}}{8} \right]_0^b$$

(from the integral tables)

$$I_y = \frac{2b}{a} \left[ \frac{x(a^2 - x^2)^{1/2}}{4} + \frac{a^2 \sqrt{a^2 - x^2}}{8} \right]_0^b$$

$$I_y = \frac{2b}{a} \right[ \frac{a^2 \sqrt{a^2 - x^2}}{4} - \frac{a^2 \sqrt{a^2 - x^2}}{8} \right]_0^b$$

$$I_y = \frac{2b a^2 \pi}{8}$$

$$I_y = \frac{2a^3 b n}{8}$$

Evaluating, we get

$I_y = 49.09 \text{ m}^4$

The area of the ellipse (half ellipse) is

$$A = 2 \int_0^b \int_x^x \frac{x}{\sqrt{b^2 - x^2}} \, dy \, dx$$

$$A = 7.85 \text{ m}^2$$

Finally

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{49.09}{7.85}}$$

$k_y = 2.5$
Problem 8.23  What are the values of $I_x$ and $k_y$ for the elliptical area of the airplane's wing in Problem 8.22?

Solution:

$I_x = \int y^2 \, dA = 2 \int_0^a \int_0^{\sqrt{a^2 - y^2}} y^2 \, dx \, dy$

$I_x = 2 \int_0^a \left[ \int_0^{\sqrt{a^2 - y^2}} y^2 \, dx \right] \, dy$

$I_x = 2 \int_0^a \left[ \frac{b^3}{3a^2} (a^2 - x^2)^{3/2} \right]_0^a \, dy$

$I_x = \frac{2b^3}{3a^2} \int_0^a \left[ \frac{a^2}{4} + \frac{3a^2 \sqrt{a^2 - x^2}}{8} + \frac{3a^2 \pi}{2} \right] \, dy$

$I_x = \frac{2b^3}{3a^2} \left[ \frac{3a^2}{8} \sin^{-1} \left( \frac{x}{a} \right) \right]_0^a$

$I_x = \frac{2b^3}{3a^2} \left[ \frac{3a^2}{8} \left( \frac{\pi}{2} \right) \right]$

$I_x = \frac{2b^3}{27} \left( \frac{\pi}{8} \right) a \left( \frac{\pi}{2} \right)$

$I_x = \frac{2ab^3\pi}{2} = \frac{ab^3\pi}{8}$

Evaluating ($a = 5$, $b = 1$)

$I_x = \frac{5\pi}{8} = 1.96 \text{ m}^4$

From Problem 8.22, the area of the wing is $A = 7.85 \text{ m}^2$

$k_y = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{1.96}{7.85}} \approx 0.500 \text{ m}$

Problem 8.24  Determine $I_y$ and $k_y$.

Solution:  The straight line and curve intersect where $x = x^2 - 20$.  
Solving this equation for $x$, we obtain

$x = \frac{1 \pm \sqrt{1 + 80}}{2} = -4, 5$

If we use a vertical strip: the area

$dA = [x - (x^2 - 20)] \, dx$.

Therefore

$I_y = \int_A x^3 \, dA = \int_{-4}^5 x^3 (x - x^2 + 20) \, dx$

$= \left[ \frac{x^4}{4} - \frac{x^5}{5} + \frac{20x^3}{3} \right]_{-4}^5 = 522$.

The area is

$A = \int_A dA = \int_{-4}^5 (x - x^2 + 20) \, dx$

$= \left[ \frac{x^2}{2} - \frac{x^3}{3} + 20x \right]_{-4}^5 = 122$.

So $k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{522}{122}} = 2.07$. 

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Problem 8.25 Determine $I_x$ and $k_x$ for the area in Problem 8.24.

Solution: Let us determine the moment of inertia about the $x$ axis of a vertical strip holding $x$ and $dx$ fixed:

$$(I_x)_{strip} = \int y^2 \, dA_x = \int_{x=4}^{x=5} y^2 \, (dx \, dy) = dx \left[ \frac{y^3}{3} \right]_{x=4}^{x=5} = \frac{dx}{3} (-x^6 + 60x^4 + x^3 - 1200x^2 + 8000).$$

Integrating this value from $x = -4$ to $x = 5$ (see the solution to Problem 8.24), we obtain $I_x$ for the entire area:

$$I_x = \int_{-4}^{5} \frac{1}{3} (-x^6 + 60x^4 + x^3 - 1200x^2 + 8000) \, dx$$

$$= \left[ \frac{x^7}{21} + 4x^5 + \frac{x^4}{12} - \frac{400x^3}{3} + \frac{8000x}{3} \right]_{-4}^{5} = 10,900.$$  

From the solution to Problem 8.24, $A = 122$ so

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{10,900}{122}} = 9.45.$$

Problem 8.26 A vertical plate of area $A$ is beneath the surface of a stationary body of water. The pressure of the water subjects each element $dA$ of the surface of the plate to a force $(p_0 + \gamma y) \, dA$, where $p_0$ is the pressure at the surface of the water and $\gamma$ is the weight density of the water. Show that the magnitude of the moment about the $x$ axis due to the pressure on the front face of the plate is

$$M_{x \, axis} = p_0 y A + y I_x,$$

where $\bar{y}$ is the $y$ coordinate of the centroid of $A$ and $I_x$ is the moment of inertia of $A$ about the $x$ axis.

Solution: The moment about the $x$ axis is $dM = (p_0 + \gamma y) y \, dA$ integrating over the surface of the plate:

$$M = \int_A (p_0 + \gamma y) y \, dA.$$  

Noting that $p_0$ and $\gamma$ are constants over the area,

$$M = p_0 \int_A y \, dA + \gamma \int_A y^2 \, dA.$$  

By definition,

$$\bar{y} = \int_A y \, dA$$

and $I_x = \int_A y^2 \, dA$.

then $M = p_0 \bar{y} A + \gamma I_x$, which demonstrates the result.
Problem 8.27 Using the procedure described in Active Example 8.3, determine $I_x$ and $k_x$ for the composite area by dividing it into rectangles 1 and 2 as shown.

**Solution:** Using results from Appendix B and applying the parallel-axis theorem, the moment of inertia about the $x$ axis for area 1 is

$$I_{x1} = I_x + d^2 A = \frac{1}{12}(1 \text{ m})(3 \text{ m})^3 + (2.5 \text{ m})(1 \text{ m})(3 \text{ m})$$

$$= 21.0 \text{ m}^4$$

The moment of inertia about the $x$ axis for area 2 is

$$I_{x2} = \frac{1}{3}(3 \text{ m})(1 \text{ m})^3 = 1 \text{ m}^4.$$  

The moment of inertia about the $x$ axis for the composite area is

$$I_x = I_{x1} + I_{x2} = 22.0 \text{ m}^4.$$  

The radius of gyration about the $x$ axis is

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{22.0 \text{ m}^4}{6 \text{ m}^2}} = 1.91 \text{ m}.$$  

$I_x = 22.0 \text{ m}^4$, $k_x = 1.91 \text{ m}$.

Problem 8.28 Determine $I_y$ and $k_y$ for the composite area by dividing it into rectangles 1 and 2 as shown.

**Solution:** Using results from Appendix B, the moment of inertia about the $y$ axis for area 1 is

$$I_{y1} = \frac{1}{3}(3 \text{ m})(1 \text{ m})^3 = 1 \text{ m}^4.$$  

The moment of inertia about the $y$ axis for area 2 is

$$I_{y2} = \frac{1}{3}(1 \text{ m})(3 \text{ m})^3 = 9 \text{ m}^4.$$  

The moment of inertia about the $y$ axis for the composite area is

$$I_y = I_{y1} + I_{y2} = 10 \text{ m}^4.$$  

The radius of gyration about the $y$ axis is

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{10 \text{ m}^4}{6 \text{ m}^2}} = 1.29 \text{ m}.$$  

$I_y = 10 \text{ m}^4$, $k_y = 1.29 \text{ m}$. 

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Problem 8.29  Determine $I_x$ and $k_x$.

Solution:  Break into 3 rectangles

\[
I_x = \frac{1}{3}(0.6)(0.2)^3 + \frac{1}{12}(0.2)(0.6)^3 + (0.2)(0.6)(0.5)^2
\]
\[
+ \frac{1}{12}(0.8)(0.2)^3 + (0.8)(0.2)(0.9)^2 = 0.1653 \text{ m}^4
\]

\[
A = (0.2)(0.6) + (0.6)(0.2) + (0.8)(0.2) = 0.4 \text{ m}^2
\]

\[
k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{0.1653}{0.4}} = 0.643 \text{ m}
\]

\[
\Rightarrow I_x = 0.1653 \text{ m}^4
\]

\[
k_x = 0.643 \text{ m}
\]

Problem 8.30  In Example 8.4, determine $I_x$ and $k_x$ for the composite area.

Solution:  The area is divided into a rectangular area without the cutout (part 1), a semicircular areas without the cutout (part 2), and the circular cutout (part 3).

Using the results from Appendix B, the moment of inertia of part 1 about the $x$ axis is

\[
(I_x)_1 = \frac{1}{12}(120 \text{ mm})(80 \text{ mm})^3 = 5.12 \times 10^6 \text{ mm}^4
\]

the moment of inertia of part 2 is

\[
(I_x)_2 = \frac{1}{8}\pi(40 \text{ mm})^4 = 1.01 \times 10^6 \text{ mm}^4
\]

and the moment of inertia of part 3 is

\[
(I_x)_3 = \frac{1}{4}\pi(20 \text{ mm})^4 = 1.26 \times 10^5 \text{ mm}^4
\]

The moment of inertia of the composite area is

\[
I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 = 6.00 \times 10^6 \text{ mm}^4
\]

From Example 8.4, the composite area is $A = 1.086 \times 10^4 \text{ mm}^2$, so the radius of gyration about the $x$ axis is

\[
k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{6.00 \times 10^6}{1.086 \times 10^4}} = 23.5 \text{ mm}
\]

\[
I_x = 6.00 \times 10^6 \text{ mm}^4, k_x = 23.5 \text{ mm}
\]
Problem 8.31  Determine $I_x$ and $k_x$.

Solution: Break into 3 rectangles — See 8.29
First locate the centroid

$$d = \frac{(0.6)(0.2)(0.1) + (0.2)(0.6)(0.5) + (0.8)(0.2)(0.9)}{(0.6)(0.2) + (0.2)(0.6) + (0.8)(0.2)} = 0.54 \text{ m}$$

$$I_x = \frac{1}{12}(0.6)(0.2)^2 + 0.2(0.6)(d - 0.5)^2$$
$$+ \frac{1}{12}(0.6)(0.2)^3 + (0.6)(0.2)(d - 0.1)^2$$
$$+ \frac{1}{12}(0.8)(0.2)^3 + (0.8)(0.2)(0.9 - d)^2 = 0.0487 \text{ m}^4$$

$$k_x = \frac{I_x}{A} = \sqrt{\frac{0.0487 \text{ m}^4}{0.4 \text{ m}^2}} = 0.349 \text{ m}$$

Problem 8.32  Determine $I_y$ and $k_y$.

Solution: Break into 3 rectangles — See 8.29

$$I_y = \frac{1}{12}(0.6)(0.2)^3 + \frac{1}{12}(0.2)(0.6)^3 + \frac{1}{12}(0.2)(0.8)^3$$
$$= 0.01253 \text{ m}^4$$

$$k_y = \frac{I_y}{A} = \sqrt{\frac{0.01253 \text{ m}^4}{0.4 \text{ m}^2}} = 0.1770 \text{ m}$$

Problem 8.33  Determine $J_O$ and $k_O$.

Solution: See 8.29, 8.31 and 8.32

$$J_O = I_x + I_y = 0.0612 \text{ m}^4$$

$$k_O = \frac{J_O}{A} = \sqrt{\frac{0.0612 \text{ m}^4}{0.4 \text{ m}^2}} = 0.391 \text{ m}$$
Problem 8.34  If you design the beam cross section so that $I_x = 6.4 \times 10^5$ mm$^4$, what are the resulting values of $I_y$ and $J_O$?

Solution:  The area moment of inertia for a triangle about the base is

$$I_x = \left(\frac{1}{12}\right)bh^3,$$

from which $I_x = 2 \left(\frac{1}{12}\right) (60)h^3 = 10h^3$ mm$^4$,

$$I_x = 10h^3 = 6.4 \times 10^5$ mm$^4$,

from which $h = 40$ mm.

$$I_y = 2 \left(\frac{1}{12}\right) (2h)(30^3) = \left(\frac{1}{3}\right) h(30^3)$$

from which $I_y = \left(\frac{1}{3}\right) (40)(30^3) = 3.6 \times 10^5$ mm$^4$

and $J_O = I_x + I_y = 3.6 \times 10^5 + 6.4 \times 10^5 = 1 \times 10^6$ mm$^4$
Problem 8.35 Determine $I_y$ and $k_y$.

Solution: Divide the area into three parts:

Part (1): The top rectangle.
\[ A_1 = 160 \times 40 = 6.4 \times 10^3 \text{ mm}^2, \]
\[ d_{x1} = \frac{160}{2} = 80 \text{ mm}, \]
\[ I_{y1} = \left( \frac{1}{12} \right) (40)(160^3) = 1.3653 \times 10^7 \text{ mm}^4. \]

From which
\[ I_{y1} = d_{x1}^2 A_1 + I_{y1} = 5.4613 \times 10^7 \text{ mm}^4. \]

Part (2): The middle rectangle:
\[ A_2 = (200 - 80) \times 40 = 4.8 \times 10^3 \text{ mm}^2, \]
\[ d_{x2} = 20 \text{ mm}, \]
\[ I_{y2} = \left( \frac{1}{12} \right) (120)(40^3) = 6.4 \times 10^6 \text{ mm}^4. \]

From which
\[ I_{y2} = d_{x2}^2 A_2 + I_{y2} = 2.56 \times 10^6 \text{ mm}^4. \]

Part (3) The bottom rectangle:
\[ A_3 = 120 \times 40 = 4.8 \times 10^3 \text{ mm}^2, \]
\[ d_{x3} = \frac{120}{2} = 60 \text{ mm}, \]
\[ I_{y3} = \left( \frac{1}{12} \right) 40(120^3) = 5.76 \times 10^6 \text{ mm}^4. \]

From which
\[ I_{y3} = d_{x3}^2 A_3 + I_{y3} = 2.304 \times 10^7 \text{ mm}^4. \]

The composite:
\[ I_y = I_{y1} + I_{y2} + I_{y3} = 8.0213 \times 10^7 \text{ mm}^4 \]
\[ k_y = \left( \frac{I_y}{A_1 + A_2 + A_3} \right)^{1/2} = 70.8 \text{ mm}. \]
Problem 8.36  Determine $I_x$ and $k_x$.

Solution: Use the solution to Problem 8.35. Divide the area into three parts:

Part (1): The top rectangle.

$A_1 = 6.4 \times 10^3$ mm$^2$,

$d_{x1} = 200 - 20 = 180$ mm,

$I_{xx1} = \left(\frac{1}{12}\right)(160)(40^3) = 8.533 \times 10^3$ mm$^4$.

From which

$I_{x1} = d_{x1}^2A_1 + I_{xx1} = 2.082 \times 10^6$ mm$^4$

Part (2): The middle rectangle:

$A_2 = 4.8 \times 10^3$ mm$^2$,

$d_{x2} = \frac{120}{2} = 60$ mm,

$I_{xx2} = \left(\frac{1}{12}\right)(40)(120^3) = 5.76 \times 10^6$ mm$^4$

from which

$I_{x2} = d_{x2}^2A_2 + I_{xx2} = 5.376 \times 10^7$ mm$^4$

Part (3): The bottom rectangle:

$A_3 = 4.8 \times 10^3$ mm$^2$,

$d_{x3} = 20$ mm,

$I_{xx3} = \left(\frac{1}{12}\right)(120)(40^3) = 6.4 \times 10^5$ mm$^4$

and $I_{x3} = d_{x3}^2A_3 + I_{xx3} = 2.56 \times 10^6$ mm$^4$.

The composite:

$I_x = I_{x1} + I_{x2} + I_{x3} = 2.645 \times 10^6$ mm$^4$

$k_x = \sqrt{\frac{I_x}{A_1 + A_2 + A_3}} = 128.6$ mm

Problem 8.37  Determine $I_{xy}$.

Solution: (See figure in Problem 8.35). Use the solutions in Problems 8.35 and 8.36. Divide the area into three parts:

Part (1): $A_1 = 160(40) = 6.4 \times 10^3$ mm$^2$,

$d_{x1} = 160$ mm,

$d_{y1} = 200 - 20 = 180$ mm,

$I_{xy1} = 0$.

from which

$I_{xy1} = d_{x1}d_{y1}A_1 + I_{xy1} = 9.216 \times 10^7$ mm$^4$.

Part (2): $A_2 = (200 - 80)(40) = 4.8 \times 10^3$ mm$^2$,

$d_{x2} = 20$ mm,

$d_{y2} = \frac{120}{2} = 60$ mm,

from which

$I_{xy2} = d_{x2}d_{y2}A_2 = 9.6 \times 10^5$ mm$^4$.

Part (3): $A_3 = 120(40) = 4.8 \times 10^3$ mm$^2$,

$d_{x3} = 120$ mm,

$d_{y3} = 20$ mm,

from which

$I_{xy3} = d_{x3}d_{y3}A_3 = 5.76 \times 10^6$.

The composite:

$I_{xy} = I_{xy1} + I_{xy2} + I_{xy3} = 1.0752 \times 10^8$ mm$^4$
Problem 8.38  Determine $I_x$ and $k_x$.

Solution:  The strategy is to use the relationship $I_x = d^2A + I_w$, where $I_w$ is the area moment of inertia about the centroid. From this $I_w = -d^2A + I_x$. Use the solutions to Problems 8.35, 8.36, and 8.37. Divide the area into three parts and locate the centroid relative to the coordinate system in the Problems 8.35, 8.36, and 8.37.

Part (1) $A_1 = 6.4 \times 10^3 \text{ mm}^2$,

$\frac{160}{2} = 80 \text{ mm}, \quad d_{11} = 20 \text{ mm}$

$\frac{120}{2} = 60 \text{ mm}, \quad d_{12} = 20 \text{ mm}$

The total area is

$A = A_1 + A_2 + A_3 = 1.6 \times 10^4 \text{ mm}^2$.

The centroid coordinates are

$x = \frac{A_1d_{11} + A_2d_{12} + A_3d_{13}}{A} = 56 \text{ mm},

y = \frac{A_1d_{12} + A_2d_{12} + A_3d_{13}}{A} = 108 \text{ mm}

from which

$I_w = -y^2A + I_x = -1.866 \times 10^8 + 2.645 \times 10^8

= 7.788 \times 10^7 \text{ mm}^4

k_w = \sqrt{\frac{I_w}{A}} = 69.77 \text{ mm}$

Problem 8.39  Determine $I_y$ and $k_y$.

Solution:  The strategy is to use the relationship $I_y = d^2A + I_w$, where $I_w$ is the area moment of inertia about the centroid. From this $I_w = -d^2A + I_y$. Use the solution to Problem 8.38. The centroid coordinates are $x = 56 \text{ mm}, \quad y = 108 \text{ mm}$, from which

$I_w = -x^2A + I_y = -5.0176 \times 10^7 + 8.0213 \times 10^7

= 3.0 \times 10^7 \text{ mm}^4,

k_w = \sqrt{\frac{I_w}{A}} = 43.33 \text{ mm}$

Problem 8.40  Determine $I_{xy}$.

Solution:  Use the solution to Problem 8.37. The centroid coordinates are

$x = 56 \text{ mm}, \quad y = 108 \text{ mm},

from which $I_{xy} = -xyA + I_{xy} = -9.6768 \times 10^7 + 1.0752 \times 10^8

= 1.0752 \times 10^8 \text{ mm}^4$
Problem 8.41  Determine $I_x$ and $k_x$.

**Solution:** Divide the area into two parts:

Part (1): a triangle and Part (2): a rectangle. The area moment of inertia for a triangle about the base is

$$I_x = \left(\frac{1}{12}\right) bh^3.$$  

The area moment of inertia about the base for a rectangle is

$$I_x = \left(\frac{1}{3}\right) bh^3.$$  

Part (1) $I_{x1} = \left(\frac{1}{12}\right) 4(3^3) = 9 \text{ m}^4$.

Part (2) $I_{x2} = \left(\frac{1}{3}\right) 3(3^3) = 27$.

The composite: $I_x = I_{x1} + I_{x2} = 36 \text{ m}^4$. The area:

$$A = \left(\frac{1}{2}\right) 4(3) + 3(3) = 15 \text{ m}^2.$$  

$$k_x = \sqrt{\frac{I_x}{A}} = 1.549 \text{ m}.$$  

Problem 8.42  Determine $J_O$ and $k_O$.

**Solution:** (See Figure in Problem 8.41.) Use the solution to Problem 8.41.

Part (1): The area moment of inertia about the centroidal axis parallel to the base for a triangle is

$$I_{y1} = \left(\frac{1}{36}\right) bh^3 = \left(\frac{1}{36}\right) 3(4^3) = 5.3333 \text{ m}^4.$$  

from which

$$I_{y2} = (5.5)^2 A_2 + I_{y1} = 279 \text{ m}^4,$$

where $A_2 = 9 \text{ m}^2$.

The composite: $I_y = I_{y1} + I_{y2} = 327 \text{ m}^4$, from which, using a result from Problem 8.41,

$$J_O = I_x + I_y = 327 + 36 = 363 \text{ m}^4$$

and $k_O = \sqrt{\frac{J_O}{A}} = 4.92 \text{ m}$. 

Part (2): The area moment of inertia about a centroid parallel to the base for a rectangle is

$$I_{y2} = \left(\frac{1}{12}\right) bh^3 = \left(\frac{1}{12}\right) 3(3^3) = 6.75 \text{ m}^4.$$
Problem 8.43 Determine $I_{xy}$.

Solution: (See Figure in Problem 8.41.) Use the results of the solutions to Problems 8.41 and 8.42. The area cross product of the moment of inertia about centroidal axes parallel to the bases for a triangle is $I_{xy} = \frac{1}{12}bh^2$, and for a rectangle it is zero. Therefore:

$$I_{xy} = 18 \text{ m}^4$$

and $I_{xy} = (1.5)(5.5)A_2 = 74.25 \text{ m}^4$.

$I_{xy} = I_{xy1} + I_{xy2} = 92.25 \text{ m}^4$

Problem 8.44 Determine $I_x$ and $k_x$.

Solution: Use the results of Problems 8.41, 8.42, and 8.43. The strategy is to use the parallel axis theorem and solve for the area moment of inertia about the centroidal axis. The centroidal coordinate

$$y = \frac{A_1(1) + A_2(1.5)}{A} = 1.3 \text{ m}.$$

From which

$$I_x = -y^2A + I_x = 10.65 \text{ m}^4$$

and $k_x = \sqrt{\frac{I_x}{A}} = 0.843$ m

Problem 8.45 Determine $J_O$ and $k_O$.

Solution: Use the results of Problems 8.41, 8.42, and 8.43. The strategy is to use the parallel axis theorem and solve for the area moment of inertia about the centroidal axis. The centroidal coordinate:

$$x = \frac{A_1(\frac{8}{7}) + A_2(5.5)}{A} = 4.3667 \text{ m},$$

from which

$$I_{xy} = -x^2A + I_y = 40.98 \text{ m}^4.$$

Using a result from Problem 8.44,

$$J_O = I_{xy} + I_{xy} = 10.65 + 40.98 = 51.63 \text{ m}^4$$

and $k_O = \sqrt{\frac{J_O}{A}} = 1.855$ m

Problem 8.46 Determine $I_{xy}$.

Solution: Use the results of Problems 8.41–8.45. The strategy is to use the parallel axis theorem and solve for the area moment of inertia about the centroidal axis. Using the centroidal coordinates determined in Problems 8.44 and 8.45,

$$I_{xy} = -xyA + I_{xy} = -85.15 + 92.25 = 7.1 \text{ m}^4$$
Problem 8.47  Determine $I_x$ and $k_x$.

Solution:  Let Part 1 be the entire rectangular solid without the hole and let part 2 be the hole.

Area $\Delta h b = \pi R^2 = (80)(120) - \pi R^2$

$I_{x1} = \frac{1}{2} bh^3$ where $b = 80$ mm

$h = 120$ mm

$I_{x1} = \frac{1}{2}(80)(120)^3 = 4.608 \times 10^7$ mm$^4$

For Part 2,

$I_{x2} = \frac{1}{2} \pi R^4 = \frac{1}{4} \pi (20)^4$ mm$^4$

$I_{x2} = 1.257 \times 10^5$ mm$^4$

$I_x = I_{x2} + d^2 A$

where $A = \pi R^2 = 1257$ mm$^2$

$d = 80$ mm

$I_{x2} = 1.257 \times 10^5 + \pi (20)^2(80)^2$

$I_{x2} = 0.126 \times 10^6 + 8.042 \times 10^5$ mm$^4$

$= 8.168 \times 10^6$ mm$^4 = 0.817 \times 10^7$ mm$^4$

$I_x = I_{x1} - I_{x2} = 3.79 \times 10^7$ mm$^4$

Area $\Delta = 8343$ mm$^2$

$k_x = \sqrt{\frac{I_x}{\Delta}} = 67.4$ mm
Problem 8.48  Determine $J_O$ and $k_O$.

Solution:  For the rectangle,

$J_{O1} = I_{x1} + I_{y1} = \frac{1}{12}bh^3 + \frac{1}{12}bh^3$

$J_{O1} = 4.608 \times 10^7 + 2.048 \times 10^7 \text{ mm}^4$

$A_1 = bh = 9600 \text{ mm}^2$

For the circular cutout about $x'y'$

$J_{O2} = I_{x2} + I_{y2} = \frac{1}{4}\pi R^4 + \frac{1}{4}\pi R^4$

$J_{O2} = 1.257 \times 10^3 + 1.257 \times 10^3 \text{ mm}^4$

$J_{O2} = 2.513 \times 10^3 \text{ mm}^4$

Using the parallel axis theorem to determine $J_{O2}$ (about $x, y$)

$J_{O2} = J_{O2} + (d_x^2 + d_y^2)A_2$

$A_2 = \pi R^2 = 1257 \text{ mm}^2$

$J_{O2} = 1.030 \times 10^3 \text{ mm}^4$

$J_O = J_{O1} - J_{O2}$

$J_O = 6.656 \times 10^7 - 1.030 \times 10^7 \text{ mm}^4$

Problem 8.49  Determine $I_{xy}$.

Solution:

$A_1 = (80)(120) = 9600 \text{ mm}^2$

$A_2 = \pi R^2 = \pi(20)^2 = 1257 \text{ mm}^2$

For the rectangle ($A_1$)

$I_{xy1} = \frac{1}{12}b^2h^2 = \frac{1}{12}(80)^2(120)^2$

$I_{xy1} = 2.304 \times 10^7 \text{ mm}^2$

For the cutout

$I_{xy2} = 0$

and by the parallel axis theorem

$I_{xy2} = I_{xy2} + A_2(d_x)(d_y)$

$I_{xy2} = 0 + (1257)(40)(80)$

$I_{xy2} = 4.021 \times 10^6 \text{ mm}^4$

$I_{xy} = I_{xy1} - I_{xy2}$

$I_{xy} = 2.304 \times 10^7 - 0.402 \times 10^7 \text{ mm}^4$

$I_{xy} = 1.90 \times 10^7 \text{ mm}^4$
Problem 8.50  Determine \( I_x \) and \( k_x \).

Solution: We must first find the location of the centroid of the total area. Let us use the coordinates \( XY \) to do this. Let \( A_1 \) be the rectangle and \( A_2 \) be the circular cutout. Note that by symmetry \( X_c = 40 \text{ mm} \).

<table>
<thead>
<tr>
<th></th>
<th>Area</th>
<th>( X_c )</th>
<th>( Y_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle ( A_1 )</td>
<td>9600 mm(^2)</td>
<td>40 mm</td>
<td>60 mm</td>
</tr>
<tr>
<td>Circle ( A_2 )</td>
<td>1257 mm(^2)</td>
<td>40 mm</td>
<td>80 mm</td>
</tr>
</tbody>
</table>

\( A_1 = 9600 \text{ mm}^2 \)
\( A_2 = 1257 \text{ mm}^2 \)

For the composite,
\[ X_c = \frac{A_1 X_{c1} - A_2 X_{c2}}{A_1 - A_2} = 40 \text{ mm} \]
\[ Y_c = \frac{A_1 Y_{c1} - A_2 Y_{c2}}{A_1 - A_2} = 57.0 \text{ mm} \]

Now let us determine \( I_x \) and \( k_x \) about the centroid of the composite body.

**Rectangle** about its centroid (40, 60) mm
\[ I_{x1} = \frac{1}{12} bh^3 = \frac{1}{12} (80)(120)^3 \]
\[ I_{x1} = 1.152 \times 10^7 \text{ mm}^4 \]

Now to \( C \)
\[ I_{x2} = I_{x1} + (60 - Y_c)^2 A_1 \]
\[ I_{x2} = 1.161 \times 10^7 \text{ mm}^4 \]

**Circular cutout** about its centroid
\[ A_2 = \pi R^2 = (20)^2 \pi = 1257 \text{ mm}^2 \]
\[ I_{x2} = \frac{\pi}{4} R^4 = \pi (20)^4 / 4 \]
\[ I_{x2} = 1.26 \times 10^5 \text{ mm}^4 \]

Now to \( C \rightarrow d_{c2} = 80 - 57 = 23 \text{ mm} \)
\[ I_{x2} = I_{x1} + (d_{c2})^2 A_2 \]
\[ I_{x2} = 7.91 \times 10^5 \text{ mm}^4 \]

For the composite about the centroid
\[ I_x = I_{x1} - I_{x2} \]
\[ I_x = 1.08 \times 10^7 \text{ mm}^4 \]

The composite Area = \( 9600 - 1257 = 8343 \text{ mm}^2 \)
\[ k_x = \sqrt{\frac{I_x}{A}} = 36.0 \text{ mm} \]
Problem 8.51  Determine $I_y$ and $k_y$.

Solution: From the solution to Problem 8.50, the centroid of the composite area is located at (40, 57.0) mm.

The area of the rectangle, $A_1$, is 9600 mm$^2$.

The area of the cutout, $A_2$, is 1257 mm$^2$.

The area of the composite is 8343 mm$^2$.

1. Rectangle about its centroid (40, 60) mm.
   
   \[ I_{y1} = \frac{1}{12}bh^3 = \frac{1}{12}(120)(80)^3 \]
   
   \[ I_{y1} = 5.12 \times 10^6 \text{ mm}^4 \]
   
   \[ d_{y1} = 0 \]

2. Circular cutout about its centroid (40, 80)
   
   \[ I_{y2} = \pi R^4/4 = 1.26 \times 10^5 \text{ mm}^4 \]
   
   \[ d_{y2} = 0 \]

Since $d_{y1}$ and $d_{y2}$ are zero, (no translation of axes in the $x$-direction), we get

\[ I_y = I_{y1} - I_{y2} \]

\[ I_y = 4.99 \times 10^6 \text{ mm}^4 \]

Finally,

\[ k_y = \sqrt{\frac{I_y}{A_1 - A_2}} = \sqrt{\frac{4.99 \times 10^6}{8343}} \]

\[ k_y = 24.5 \text{ mm} \]

Problem 8.52  Determine $J_O$ and $k_O$.

Solution: From the solutions to Problems 8.51 and 8.52,

\[ I_x = 1.07 \times 10^7 \text{ mm}^4 \]

\[ I_y = 4.99 \times 10^6 \text{ mm}^4 \]

and $A = 8343 \text{ mm}^2$

\[ J_O = I_x + I_y = 1.57 \times 10^7 \text{ mm}^4 \]

\[ k_O = \sqrt{\frac{J_O}{A}} = 43.4 \text{ mm} \]
Problem 8.53  Determine $I_y$ and $k_y$.

Solution: Treat the area as a circular area with a half-circular cutout: From Appendix B,

\[ (I_y)_1 = \frac{1}{4} \pi (20)^4 \text{ mm}^4 \]

and \( (I_y)_2 = \frac{1}{4} \pi (12)^4 \text{ mm}^4 \).

so \( I_y = \frac{1}{4} \pi (20)^4 - \frac{1}{4} \pi (12)^4 = 1.18 \times 10^5 \text{ mm}^4 \).

The area is \( A = \pi (20)^2 - \frac{1}{4} \pi (12)^2 = 1030 \text{ mm}^2 \)

so, \( k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{1.18 \times 10^5}{1.03 \times 10^5}} \)

\( = 10.7 \text{ mm} \)

Problem 8.54  Determine $J_O$ and $k_O$.

Solution: Treating the area as a circular area with a half-circular cutout as shown in the solution of Problem 8.53, from Appendix B,

\[ (J_O)_1 = (I_y)_1 + (I_z)_1 = \frac{1}{4} \pi (20)^4 \text{ mm}^4 \]

and \( (J_O)_2 = (I_y)_2 + (I_z)_2 = \frac{1}{4} \pi (12)^4 \text{ mm}^4 \).

Therefore \( J_O = \frac{1}{4} \pi (20)^4 - \frac{1}{4} \pi (12)^4 \)

\( = 2.35 \times 10^5 \text{ mm}^4 \).

From the solution of Problem 8.53,

\[ A = 1030 \text{ in}^2 R_o = \sqrt{\frac{J_O}{A}} \]

\( = \sqrt{\frac{2.35 \times 10^5}{1.03 \times 10^5}} = 15.1 \text{ mm} \).
Problem 8.55  Determine $I_y$ and $k_y$ if $h = 3$ m.

**Solution:** Break the composite into two parts, a rectangle and a semi-circle.

For the semi-circle

\[
I_{xc} = \left(\frac{\pi}{8} - \frac{9}{8\pi}\right)R^2
\]

\[
I_{yc} = \frac{1}{8}\pi R^4
d = \frac{4R}{3\pi}
\]

To get moments about the $x$ and $y$ axes, the $(d_{xc}, d_{yc})$ for the semi-circle are

\[
d_{xc} = 0, \quad d_{yc} = 3 \text{ m} + \frac{4R}{3\pi}
\]

and $A_s = \pi R^2/2 = 2.26 \text{ m}^2$.

\[
I_{xc} = \frac{1}{8}\pi R^4
\]

and $I_{yc} = I_{xc} + d_{yc}^2 A_s$  \((d_s = 0)\)

\[
I_{yc} = \pi (1.2)^2 / 8
\]

\[
I_{yc} = 0.814 \text{ m}^4
\]

For the Rectangle

\[
I_{xR} = \frac{1}{12}bh^3
\]

\[
I_{yR} = \frac{1}{12}hb^3
\]

$A_R = bh$

$A_k = bh$

To get moments of area about the $x, y$ axes, $d_{xR} = 0, \quad d_{yR} = 1.5 \text{ m}$

\[
I_{xR} = I_{yR} + (d_{xR})^2 A_R
\]

\[
I_{yR} = \frac{1}{12}(3.4)^3 \text{ m}^4
\]

\[
I_{yR} = 3.456 \text{ m}^4
\]

$A_R = bh = 7.2 \text{ m}^2$

\[
I_y = I_{yc} + I_{yR}
\]

\[
I_y = 4.27 \text{ m}^4
\]

To find $k_y$, we need the total area, $A = A_k + A_c$

\[
A = 7.20 + 2.26 \text{ m}^2
\]

\[
A = 9.46 \text{ m}^2
\]

\[
k_y = \frac{I_y}{A} = 0.672 \text{ m}
\]
Problem 8.56  Determine $I_x$ and $k_x$ if $h = 3$ m.

Solution:  Break the composite into two parts, the semi-circle and the rectangle. From the solution to Problem 8.55,

$$I_{xc} = \left(\frac{\pi}{8} - \frac{9}{8\pi}\right) R^4$$

$$d_{xc} = \left(3 + \frac{4R}{\pi}\right)\text{ m}$$

$A_c = 2.26\text{ m}^2$

$I_{xc} = I_{c/c} + A_c d_{xc}^2$

Substituting in numbers, we get

$$I_{c/c} = 0.0717\text{ m}^4$$

$$d_{xc} = 3.509\text{ m}$$

and $I_{xc} = I_{c/c} + A_c d_{xc}^2$

$I_{xc} = 27.928\text{ m}^2$

For the Rectangle $h = 3$ m, $b = 2.4$ m

Area: $A_R = bh = 7.20\text{ m}^2$

$$I_{cR} = \frac{1}{12}bh^3$$

$A_{cR} = 1.5\text{ m}$

$A_{cR} = I_{cR} + d_{cR}^2 A_{cR}$

Substituting, we get

$I_{cR} = 5.40\text{ m}^4$

$I_{cR} = 21.6\text{ m}^4$

For the composite,

$I_x = I_{cR} + I_{xc}$

$I_x = 49.5\text{ m}^4$

Also $k_x = \sqrt{\frac{I_x}{A_R + A_c}} = 2.29\text{ m}$

$k_x = 2.29\text{ m}$
Problem 8.57  If \( I_y = 5 \text{ m}^4 \), what is the dimension of \( h \)?

**Solution:** From the solution to Problem 8.55, we have:

For the semicircle

\[ I_{yc} = I_y = \pi (1.2)^{3/2} = 0.814 \text{ m}^2 \]

For the rectangle

\[ I_{yR} = I_{yR} = \frac{1}{12} (h)(2.4)^3 \text{ m}^4 \]

Also, we know \( I_{yR} + I_{yc} = 5 \text{ m}^4 \).

Hence \( 0.814 + \frac{1}{12} (h)(2.4)^3 = 5 \)

Solving, \( h = 3.63 \text{ m} \)

---

Problem 8.58  Determine \( I_y \) and \( k_y \).

**Solution:** Let the area be divided into parts as shown. The areas and the coordinates of their centroids are

\[ A_1 = (40)(50) = 2000 \text{ cm}^2, \quad \tau_1 = 25 \text{ cm}, \quad \tau_1 = 20 \text{ cm}, \]

\[ A_2 = (20)(30) = 600 \text{ cm}^2, \quad \tau_2 = 10 \text{ cm}, \quad \tau_2 = 55 \text{ cm}, \]

\[ A_3 = \frac{1}{4} \pi (30)^2 = 707 \text{ cm}^2, \]

\[ \tau_3 = 20 + \frac{4(30)}{3\pi} = 32.7 \text{ cm}, \quad \tau_3 = 40 + \frac{4(30)}{3\pi} = 52.7 \text{ cm}. \]

Using the results from Appendix B, the moments of inertia of the parts about the \( y \) axis are

\[ (I_y)_1 = \frac{1}{3} (40 \text{ cm})(50 \text{ cm})^3 = 167 \times 10^4 \text{ cm}^4, \]

\[ (I_y)_2 = \frac{1}{3} (30 \text{ cm})(20 \text{ cm})^3 = 8.00 \times 10^4 \text{ cm}^4, \]

\[ (I_y)_3 = \left( \frac{\pi}{16} - \frac{4}{9\pi} \right) (30 \text{ cm})^4 + \left[ 20 \text{ cm} + \frac{4(30 \text{ cm})}{3\pi} \right]^2 \left[ \frac{\pi}{4} (30 \text{ cm})^2 \right] \]

\[ = 80.2 \times 10^4 \text{ cm}^4. \]

The moment of inertia of the composite area about the \( y \) axis is

\[ I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 = 2.55 \times 10^6 \text{ cm}^4. \]

The composite area is \( A = A_1 + A_2 + A_3 = 3310 \text{ cm}^2. \)

The radius of gyration about the \( y \) axis is \( k_y = \sqrt{\frac{I_y}{A}} = 27.8 \text{ cm.} \)

\[ I_y = 2.55 \times 10^6 \text{ cm}^4, \quad k_y = 27.8 \text{ cm}. \]
**Problem 8.59** Determine $I_x$ and $k_x$.

**Solution:** See the solution to Problem 8.58.

Let the area be divided into parts as shown. The areas and the coordinates of their centroids are

- $A_1 = 40 \times 50 = 2000 \text{ cm}^2$, $x_1 = 25 \text{ cm}$, $y_1 = 20 \text{ cm}$,
- $A_2 = 20 \times 30 = 600 \text{ cm}^2$, $x_2 = 10 \text{ cm}$, $y_2 = 55 \text{ cm}$,
- $A_3 = \frac{1}{4} \pi (30)^2 = 707 \text{ cm}^2$, $x_3 = 20 + \frac{4(30)}{3\pi} = 32.7 \text{ cm}$,

Using the results from Appendix B, the moments of inertia of the parts about the $x$ axis are

- $(I_x)_1 = \frac{1}{3} (50 \text{ cm})(40 \text{ cm})^3 = 1.07 \times 10^6 \text{ cm}^4$,
- $(I_x)_2 = \frac{1}{12} (20 \text{ cm})(30 \text{ cm})^3 + (55 \text{ cm})^2 (600 \text{ cm}^2) = 1.86 \times 10^6 \text{ cm}^4$,
- $(I_x)_3 = \left( \frac{\pi}{4} - \frac{4}{9\pi} \right) (30 \text{ cm})^4 + \left[ 40 \text{ cm} + \frac{4(30 \text{ cm})}{3\pi} \right]^2 \left[ \frac{\pi}{4} (30 \text{ cm})^2 \right] = 2.01 \times 10^6 \text{ cm}^4$.

The moment of inertia of the composite area about the $x$ axis is

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 = 4.94 \times 10^6 \text{ cm}^4.$$ 

The composite area is $A = A_1 + A_2 + A_3 = 3310 \text{ cm}^2$.

The radius of gyration about the $y$ axis is $k_x = \sqrt{\frac{I_x}{A}} = 38.6 \text{ cm}$.

$$I_x = 4.94 \times 10^6 \text{ cm}^4, k_x = 38.6 \text{ cm}$$
Problem 8.60  Determine $I_{xy}$.

**Solution:**  See the solution to Problem 8.58.

Let the area be divided into parts as shown. The areas and the coordinates of their centroids are

$A_1 = 40 \times 50 = 2000 \text{ cm}^2$,  $\tau_1 = 25 \text{ cm}$,  $\gamma_1 = 20 \text{ cm}$,

$A_2 = 20 \times 30 = 600 \text{ cm}^2$,  $\tau_2 = 10 \text{ cm}$,  $\gamma_2 = 55 \text{ cm}$,

$A_3 = \frac{1}{3} \pi (30)^2 = 707 \text{ cm}^2$,  $\tau_3 = 20 + \frac{4(30)}{3\pi} = 32.7 \text{ cm}$,

$\gamma_3 = 40 + \frac{4(30)}{3\pi} = 52.7 \text{ cm}$.

Using the results from Appendix B, the products of inertia of the parts about are

$(I_{xy})_1 = \frac{1}{3} (50 \text{ cm})^2 (40 \text{ cm})^2 = 10.0 \times 10^4 \text{ cm}^4$,

$(I_{xy})_2 = (10 \text{ cm})(55 \text{ cm})(600 \text{ cm}^2) = 3.30 \times 10^5 \text{ cm}^4$,

$(I_{xy})_3 = \left( \frac{1}{8} \frac{4}{9\pi} \right) (30 \text{ cm})^4 + \left[ 20 \text{ cm} + \frac{4(30) \text{ cm}}{3\pi} \right] \left[ 40 \text{ cm} + \frac{4(30) \text{ cm}}{3\pi} \right] [707 \text{ cm}^2]

= 12.1 \times 10^5 \text{ cm}^4$.

The product of inertia of the composite area is

$I_{xy} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3 = 2.54 \times 10^6 \text{ cm}^4$.

$I_{xy} = 2.54 \times 10^6 \text{ cm}^4$.

Problem 8.61  Determine $I_y$ and $k_y$.

**Solution:**  See the solution to Problem 8.58.

In terms of the coordinate system used in Problem 8.58, the areas and the coordinates of their centroids are

$A_1 = 40 \times 50 = 2000 \text{ cm}^2$,  $\tau_1 = 25 \text{ cm}$,  $\gamma_1 = 20 \text{ cm}$,

$A_2 = 20 \times 30 = 600 \text{ cm}^2$,  $\tau_2 = 10 \text{ cm}$,  $\gamma_2 = 55 \text{ cm}$,

$A_3 = \frac{1}{3} \pi (30)^2 = 707 \text{ cm}^2$,  $\tau_3 = 20 + \frac{4(30)}{3\pi} = 32.7 \text{ cm}$,

$\gamma_3 = 40 + \frac{4(30)}{3\pi} = 52.7 \text{ cm}$.

The composite area is $A = A_1 + A_2 + A_3 = 3310 \text{ cm}^2$.

The $x$ coordinate of its centroid is $\bar{x} = \frac{\tau_1 A_1 + \tau_2 A_2 + \tau_3 A_3}{A} = 23.9 \text{ cm}$.

The moment of inertia about the $y$ axis in terms of the coordinate system used in Problem 8.58 is $I_y = 2.55 \times 10^6 \text{ cm}^4$. Applying the parallel axis theorem, the moment of inertia about the $y$ axis through the centroid of the area is

$I_y = 2.55 \times 10^6 \text{ cm}^4 - (23.9 \text{ cm})^2 (3310 \text{ cm}^2) = 6.55 \times 10^5 \text{ cm}^4$.

The radius of gyration about the $y$ axis is $k_y = \sqrt{\frac{I_y}{A}} = 14.1 \text{ cm}$.

$I_y = 6.55 \times 10^5 \text{ cm}^4$,  $k_y = 14.1 \text{ cm}$.
Problem 8.62 Determine $I_x$ and $k_x$.

Solution: See the solution to Problem 8.59.
In terms of the coordinate system used in Problem 8.59, the areas and the coordinates of their centroids are

\[ A_1 = (40)(50) = 2000 \text{ cm}^2, \quad \bar{x}_1 = 25 \text{ cm}, \quad \bar{y}_1 = 20 \text{ cm}, \]

\[ A_2 = (20)(30) = 600 \text{ cm}^2, \quad \bar{x}_2 = 10 \text{ cm}, \quad \bar{y}_2 = 55 \text{ cm}, \]

\[ A_3 = \frac{1}{4}(30)^2 = 707 \text{ cm}^2, \quad \bar{x}_3 = 20 + \frac{4(30)}{3\pi} = 32.7 \text{ cm}, \]

\[ \bar{y}_3 = 40 + \frac{4(30)}{3\pi} = 52.7 \text{ cm}. \]

The composite area is $A = A_1 + A_2 + A_3 = 3310 \text{ cm}^2$.

The moment of inertia about the $x$ axis is $I_x = 4.94 \times 10^6 \text{ cm}^4$.

The radius of gyration about the $x$ axis is $k_x = \sqrt{\frac{I_x}{A}} = 19.5 \text{ cm}$.

Problem 8.63 Determine $I_{xy}$.

Solution: See the solution to Problem 8.60.

\[ I_{xy} = \left[ 0 + (1.0 \text{ m})(0.8 \text{ m})(d_x - 0.5 \text{ m})(d_y - 0.4 \text{ m}) \right] \]

\[ - \left[ 0 + \pi(0.2 \text{ m})^2(d_x - 0.4 \text{ m})(d_y - 0.3 \text{ m}) \right] \]

\[ + \left[ \frac{1}{24}(0.8 \text{ m})^2(0.6 \text{ m})^2 - \frac{1}{2}(0.8 \text{ m})(0.6 \text{ m})(0.2 \text{ m}) \left( \frac{0.8 \text{ m}}{3} \right) \right] \]

\[ + \frac{1}{2}(0.8 \text{ m})(0.6 \text{ m})(d_x - 1.2 \text{ m}) \left( d_y - \frac{0.8 \text{ m}}{3} \right) \]

Solving: $I_{xy} = -0.0230 \text{ m}^4$

Check using the noncentroidal product of inertia from Problem 8.60 we have

\[ I_{xy} = I_{xy}' - Ad_xd_y = 0.2185 \text{ m}^4 - (0.914 \text{ m}^2)(0.697 \text{ m})(0.379 \text{ m}) \]

\[ = -0.0230 \text{ m}^4 \]
Problem 8.64  Determine $I_y$ and $k_y$.

Solution:  Divide the area into three parts:

Part (1) The rectangle 18 by 18 mm; Part (2) The triangle with base 6 mm and altitude 18 mm; Part (3) The semicircle of 9 mm radius.

Part (1): $A_1 = 18(18) = 324 \text{ mm}^2$.
\[ x_1 = 9 \text{ mm}, \quad y_1 = 9 \text{ mm}, \]
\[ I_{xx1} = \frac{1}{12} 18(18^3) = 8748 \text{ mm}^4, \]
\[ I_{yy1} = \frac{1}{12} 18(18^3) = 8748 \text{ mm}^4. \]

Part (2): $A_2 = \left(\frac{1}{2}\right) 18(6) = 54 \text{ mm}^2$.
\[ x_2 = 9 \text{ mm}, \quad y_2 = \frac{1}{3} 18 = 6 \text{ mm}, \]
\[ I_{xx2} = \frac{1}{12} 6(18^3) = 972 \text{ mm}^4, \]
\[ I_{yy2} = \frac{1}{18} 18(3^3) = 27 \text{ mm}^4. \]

Part (3) $A_3 = \frac{\pi(9^2)}{2} = 127.23 \text{ mm}^2$.
\[ x_3 = 9 \text{ mm}, \]
\[ y_3 = 18 + \left(\frac{4(9)}{3\pi}\right) = 21.82 \text{ mm}, \]
\[ I_{xx3} = \frac{1}{8}\pi(9^4) - \left(\frac{4(9)}{3\pi}\right)^2 A_3 = 720.1 \text{ mm}^4, \]
\[ I_{yy3} = \frac{1}{8}\pi(9^4) = 2576.5 \text{ mm}^4. \]

The composite area:
\[ A = \sum_{i=1}^{3} A_i = 397.23 \text{ mm}^2. \]

The area moment of inertia:
\[ I_y = k^2 A_1 + I_{yy1} - k^2 A_2 - I_{yy2} + k_3^2 A_3, \]
\[ I_y = 4.347 \times 10^4 \text{ mm}^4, \]
\[ k_y = \sqrt{\frac{I_y}{A}} = 10.461 \text{ mm}. \]
**Problem 8.65**  Determine \( I_x \) and \( k_x \).

**Solution:** Use the results of the solution to Problem 8.64.

\[
I_x = x_1^2 A_1 + x_2^2 A_2 - x_1 x_2 A_1 + y_3^2 A_3 + I_{xx3},
\]
\[
I_x = 9.338 \times 10^4 \text{ mm}^4,
\]
\[
k_x = \frac{I_x}{A} = 15.33 \text{ mm}
\]

**Problem 8.66**  Determine \( I_{xy} \).

**Solution:** Use the results of the solutions to Problems 8.63 and 8.64.

\[
I_{xy} = x_1 y_1 A_1 + x_2 y_2 A_2 + x_3 y_3 A_3
\]
\[
I_{xy} = 4.8313 \times 10^4 \text{ mm}^4
\]

**Problem 8.67**  Determine \( I_y \) and \( k_y \).

**Solution:** We divide the composite area into a triangle (1), rectangle (2), half-circle (3), and circular cutout (4):

Triangle:

\[
(I_y)_1 = \frac{1}{2}(12)(8)^3 = 1536 \text{ cm}^4
\]

Rectangle:

\[
(I_y)_2 = \frac{1}{2}(12)(8)^3 + (12)^2(8)(12) = 14,336 \text{ cm}^4
\]

Half-Circle:

\[
(I_y)_3 = \left( \frac{\pi}{8} \frac{8}{9\pi} \right) (6)^4 + \left[ 16 + \frac{4(6)}{3\pi} \right]^2 \frac{1}{2} \pi(6)^2 = 19,593 \text{ cm}^4
\]

Circular cutout:

\[
(I_y)_4 = \frac{4}{3}\pi(2)^4 + (16)^2\pi(2)^2 = 3230 \text{ cm}^4
\]

Therefore

\[
I_y = (I_y)_1 + (I_y)_2 + (I_y)_3 - (I_y)_4 = 3.224 \times 10^4 \text{ cm}^4
\]

The area is

\[
A = A_1 + A_2 + A_3 - A_4
\]

\[
= \frac{1}{2}(12)(8) + (8)(12) + \frac{1}{2}\pi(6)^2 - \pi(2)^2 = 188 \text{ cm}^2
\]

so

\[
k_y = \frac{I_y}{A} = \frac{3.224 \times 10^4}{188} = 13.1 \text{ cm}
\]
Problem 8.68  Determine $J_O$ and $k_O$.

Solution:  $I_x$ is determined in the solution to Problem 8.67. We will determine $I_x$ and use the relation $J_O = I_x + I_y$. Using the figures in the solution to Problem 8.67,

Triangle:

$(I_x)_1 = \frac{1}{12}(8)(12)^3 = 1152$ cm$^4$.

Rectangle:

$(I_x)_2 = \frac{1}{4}(8)(12)^3 = 4608$ cm$^4$.

Half Circle:

$(I_x)_3 = \frac{1}{4}\pi(6)^4 + (6)^2 \frac{1}{4}\pi(6)^2 = 2545$ cm$^4$.

Circular Cutout:

$(I_x)_4 = \frac{1}{4}\pi(2)^4 + (6)^2 \pi (2)^2 = 465$ cm$^4$.

Therefore

$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 - (I_x)_4 = 7840$ cm$^4$.

Using the solution of Problem 8.67,

$J_O = I_x + I_y = 0.784 \times 10^5 + 3.224 \times 10^5 = 4.01 \times 10^5$ cm$^4$.

From the solution of Problem 8.67, $A = 188$ cm$^2$, so

$$R_0 = \sqrt{\frac{J_O}{A}} = \sqrt{\frac{4.01 \times 10^5}{188}} = 14.6 \text{ cm.}$$
Problem 8.69  Determine $I_x$ and $k_y$.

Solution:  Divide the area into four parts: Part (1) The rectangle 8 cm by 16 cm. Part (2): The rectangle 4 cm by 8 cm. Part (3) The semicircle of radius 4 cm, and Part (4) The circle of radius 2 cm.

Part (1): $A_1 = 16(8) = 128 \text{ cm}^2$.

\[ x_1 = 8 \text{ cm}, \]
\[ y_1 = 4 \text{ cm}, \]
\[ I_{xx1} = \frac{1}{12} \times 16(8^3) = 682.67 \text{ cm}^4, \]
\[ I_{yy1} = \frac{1}{2} \times 8(16^3) = 2730.7 \text{ cm}^4. \]

Part (2): $A_2 = 4(8) = 32 \text{ cm}^2$.

\[ x_2 = 12 \text{ cm}, \]
\[ y_2 = 10 \text{ cm}, \]
\[ I_{xx2} = \frac{1}{12} \times 8(4^3) = 42.667 \text{ cm}^4, \]
\[ I_{yy2} = \frac{1}{12} \times 4(8^3) = 170.667 \text{ cm}^4. \]

Part (3): $A_3 = \frac{\pi(4^3)}{2} = 25.133 \text{ cm}^2$.

\[ x_3 = 12 \text{ cm}, \]
\[ y_3 = 12 + \left(\frac{4(4)}{3\pi}\right) = 13.698 \text{ cm}. \]

The area moments of inertia about the centroid of the semicircle are

\[ I_{yy3} = \frac{1}{8} \pi(4^4) = 100.53 \text{ cm}^4, \]
\[ I_{xx3} = \left(\frac{1}{8} \pi(4^4) - \left(\frac{4(4)}{3\pi}\right)^2\right) A_3 = 28.1 \text{ cm}^4. \]

Check:

\[ I_{xx3} = 0.1098(R^4) = 28.1 \text{ cm}^4. \]

\[ \text{check.} \]

Part (4): $A_4 = \pi(2^2) = 12.566 \text{ cm}^2$.

\[ x_4 = 12 \text{ cm}, \]
\[ y_4 = 12 \text{ cm}, \]
\[ I_{xx4} = \left(\frac{1}{4}\right) \pi(2^4) = 12.566 \text{ cm}^4, \]
\[ I_{yy4} = I_{xx4} = 12.566 \text{ cm}^4. \]

The composite area:

\[ A = \sum_i A_i = 172.566 \text{ cm}^2. \]

The area moment of inertia:

\[ I_y = x_1^2 A_1 + x_2^2 A_2 + x_3^2 A_3 + x_4^2 A_4 - I_{yy4} \]
\[ I_y = 1.76 \times 10^4 \text{ cm}^4, \]
\[ k_y = \sqrt{\frac{I_y}{A}} = 10.1 \text{ cm}. \]

Problem 8.70  Determine $I_x$ and $k_x$.

Solution:  Use the results in the solution to Problem 8.69.

\[ I_x = y_1^2 A_1 + I_{xx1} + y_2^2 A_2 + I_{xx2} + y_3^2 A_3 + I_{xx3} - y_4^2 A_4 - I_{xx4} \]
\[ I_x = 8.89 \times 10^3 \text{ cm}^4. \]
\[ k_x = \sqrt{\frac{I_x}{A}} = 7.18 \text{ cm}. \]
Problem 8.71  Determine $I_{xy}$.

Solution: Use the results in the solution to Problem 8.69.

\[ I_{xy} = x_1y_1A_1 + x_2y_2A_2 + x_3y_3A_3 - x_4y_4A_4, \]

\[ I_{xy} = 1.0257 \times 10^4 \text{ cm}^4 \]

Problem 8.72  Determine $I_y$ and $k_y$.

Solution: Use the results in the solutions to Problems 8.69 to 8.71. The centroid is

\[ x = \frac{x_1A_1 + x_2A_2 + x_3A_3 - x_4A_4}{A} = \frac{1024 + 384 + 301.6 - 150.8}{172.567} = 9.033 \text{ cm}, \]

from which

\[ I_{xy} = -x^2A + I_y = -1.408 \times 10^4 + 1.7598 \times 10^4 = 351.8 \text{ cm}^4 \]

\[ k_y = \sqrt{\frac{I_y}{A}} = 4.52 \text{ cm} \]

Problem 8.73  Determine $I_x$ and $k_x$.

Solution: Use the results in the solutions to Problems 8.69 to 8.71. The centroid is

\[ y = \frac{y_1A_1 + y_2A_2 + y_3A_3 - y_4A_4}{A} = 5.942 \text{ cm}, \]

from which

\[ I_{xy} = -y^2A + I_x = -6092.9 + 8894 = 2801 \text{ cm}^4 \]

\[ k_x = \sqrt{\frac{I_x}{A}} = 4.03 \text{ cm} \]

Problem 8.74  Determine $I_{xy}$.

Solution: Use the results in the solutions to Problems 8.69–8.71.

\[ I_{xy} = -xyA + I_{xy} = -9.263 \times 10^3 + 1.0257 \times 10^4 = 994.5 \text{ cm}^4 \]
Problem 8.75  Determine \( I_y \) and \( k_y \).

Solution:  We divide the area into parts as shown:

\[
I_y = \frac{1}{12}(30 + 15 + 30)(30)^3 = 180,000 \text{ mm}^4
\]

\[
I_y = \frac{1}{12}(30)(10)^3 + (20)^2(10)(30)
\]

\[
= 122,500 \text{ mm}^4
\]

\[
I_y = (I_y)_s + (I_y)_t = \left( \frac{\pi}{8} - \frac{8}{9\pi} \right)(15)^4
\]

\[
+ \left[ 25 + \frac{4(15)}{3\pi} \right] \frac{1}{2}(15)^2 = 353,274 \text{ mm}^4
\]

\[
I_y = (I_y)_s = (I_y)_t = \frac{1}{4}(5\pi)^4 + (25)^2\pi(5)^2 = 49,578 \text{ mm}^4.
\]

Therefore,

\[
I_y = (I_y)_s + 3(I_y)_t + 3(I_y)_h - 3(I_y)_h = 1.46 \times 10^6 \text{ mm}^4
\]

The area is

\[
A = A_1 + 3A_2 + 3A_3 - 3A_4
\]

\[
= (30)(80) + 3(10)(30) + 3 \left( \frac{1}{4} \right) \pi(15)^2 - 3\pi(5)^2
\]

\[
= 4,125 \text{ mm}^2
\]

so \( k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{1.46 \times 10^6}{4,125}} = 18.8 \text{ mm} \)

Problem 8.76  Determine \( J_O \) and \( k_O \).

Solution:  \( I_y \) is determined in the solution to Problem 8.75.  We will determine \( I_x \) and use the relation \( J_O = I_x + I_y \).  Dividing the area as shown in the solution to Problem 8.75, we obtain

\[
I_x = \frac{1}{12}(30)(80)^3 + (25)^2(30)(80) = 2,780,000 \text{ mm}^4
\]

\[
I_x = \frac{1}{12}(10)(30)^3 + (50)^2(10)(30) = 772,500 \text{ mm}^4
\]

\[
I_x = (I_x)_s = \frac{1}{12}(10)(30)^3 = 22,500 \text{ mm}^4
\]

\[
I_x = \frac{1}{8}(5\pi)^4 + (50)^2\frac{1}{2}\pi(15)^2 = 903,453 \text{ mm}^4
\]

\[
I_x = (I_x)_s = \frac{1}{8}(5\pi)^4 = 19,880 \text{ mm}^4
\]

\[
I_x = \frac{1}{4}(5\pi)^4 + \pi(5)^2(50)^2.
\]

\[
I_x = (I_x)_t = \frac{1}{4}(5\pi)^4 = 491 \text{ mm}^4.
\]

Therefore

\[
I_x = (I_x)_s + (I_x)_t + 2(I_x)_h + (I_x)_s = 2(I_x)_h - 2(I_x)_h
\]

\[
= 4.34 \times 10^6 \text{ mm}^4
\]

and \( J_O = I_x + I_y = 5.80 \times 10^6 \text{ mm}^4 \).

From the solution to Problem 8.75, \( A = 4,125 \text{ mm}^2 \)

so \( k_O = \sqrt{\frac{J_O}{A}} \)

\[
= \sqrt{\frac{5.80 \times 10^6}{4,125}}
\]

\[
= 37.5 \text{ mm}.
\]
Problem 8.77  Determine $I_x$ and $I_y$ for the beam’s cross section.

Solution: Use the symmetry of the object

\[
\begin{align*}
I_x & = \frac{1}{12} (3 \text{ cm})(8 \text{ cm})^3 + \frac{1}{12} (3 \text{ cm})(3 \text{ cm})^3 + (3 \text{ cm})(11.5 \text{ cm})^2 \\
& + \left[ \frac{\pi (5 \text{ cm})^4}{16} - \frac{\pi (5 \text{ cm})^2}{4} \left( \frac{4(5 \text{ cm})}{3\pi} \right)^2 \\
& + \frac{\pi (5 \text{ cm})^2}{4} \left( \frac{8 \text{ cm} + 4(5 \text{ cm})}{3\pi} \right)^2 \right] \\
& - \left[ \frac{\pi (2 \text{ cm})^4}{16} - \frac{\pi (2 \text{ cm})^2}{4} \left( \frac{4(2 \text{ cm})}{3\pi} \right)^2 \\
& + \frac{\pi (2 \text{ cm})^2}{4} \left( \frac{8 \text{ cm} + 4(2 \text{ cm})}{3\pi} \right)^2 \right]
\end{align*}
\]

Solving we find

\[ I_x = 7016 \text{ cm}^4 \]

\[
\begin{align*}
I_y & = \frac{1}{12} (3 \text{ cm})(3 \text{ cm})^3 + \frac{1}{12} (8 \text{ cm})(3 \text{ cm})^3 + (8 \text{ cm})(3 \text{ cm})(6.5 \text{ cm})^2 \\
& + \left[ \frac{\pi (5 \text{ cm})^4}{16} - \frac{\pi (5 \text{ cm})^2}{4} \left( \frac{4(5 \text{ cm})}{3\pi} \right)^2 \\
& + \frac{\pi (5 \text{ cm})^2}{4} \left( 3 \text{ cm} + \frac{4(5 \text{ cm})}{3\pi} \right)^2 \right] \\
& - \left[ \frac{\pi (2 \text{ cm})^4}{16} - \frac{\pi (2 \text{ cm})^2}{4} \left( \frac{4(2 \text{ cm})}{3\pi} \right)^2 \\
& + \frac{\pi (2 \text{ cm})^2}{4} \left( 3 \text{ cm} + \frac{4(2 \text{ cm})}{3\pi} \right)^2 \right]
\end{align*}
\]

Solving we find

\[ I_y = 3122 \text{ cm}^4 \]

Problem 8.78  Determine $I_x$ and $I_y$ for the beam’s cross section.

Solution: Use Solution 8.77 and 7.39. From Problem 7.39 we know that

\[
\begin{align*}
\gamma & = 7.48 \text{ cm}, \quad A = 98.987 \text{ cm}^2 \\
I_x & = 7016 \text{ cm}^4 - A\gamma^2 = 1471 \text{ cm}^4 \\
I_y & = 3122 \text{ cm}^4 - A(0)^2 = 3122 \text{ cm}^4
\end{align*}
\]
Problem 8.79 The area $A = 2 \times 10^4$ mm$^2$. Its moment of inertia about the $y$ axis is $I_y = 3.2 \times 10^6$ mm$^4$. Determine its moment of inertia about the $\hat{y}$ axis.

Solution: Use the parallel axis theorem. The moment of inertia about the centroid of the figure is

$$I_c = -x^2A + I_y = -(120^2)(2 \times 10^4) + 3.2 \times 10^6$$

$$= 3.2 \times 10^7$$ mm$^4$.

The moment of inertia about the $\hat{y}$ axis is

$$I_\hat{y} = x^2A + I_c$$

$$I_\hat{y} = (200^2)(2 \times 10^4) + 3.2 \times 10^7$$

$$= 1 \times 10^9$$ mm$^4$


Problem 8.80 The area $A = 100$ mm$^2$ and it is symmetric about the $x'$ axis. The moments of inertia $I_x = 420$ mm$^4$, $I_y = 580$ mm$^4$, $I_O = 11000$ mm$^4$, and $I_{xy} = 4800$ mm$^4$. What are $I_x$ and $I_y$?

Solution: The basic relationships:

(1) $I_x = x^2A + I_{xc}$,
(2) $I_y = y^2A + I_{yc}$,
(3) $I_O = Ax^2 + I_x$,
(4) $I_O = I_x + I_y$,
(5) $I_x = I_{xc} + I_{xy}$, and
(6) $I_{xy} = Axy + I_{xy}$,

where the subscript $c$ applies to the primed axes, and the others to the unprimed axes. The $x$, $y$ values are the displacement of the primed axes from the unprimed axes. The steps in the demonstration are:

(i) From symmetry about the $x_c$ axis, the product of inertia $I_{xc} = 0$.
(ii) From (3): $x^2 = \frac{I_O}{A} = 100$ mm$^2$, from which $x^2 = 100$ mm$^2$.
(iii) From (6) and $I_{xy} = 0$, $y = \frac{I_{xy}}{Ax}$, from which $x^2y^2 = x^4 + \left(\frac{I_{xy}}{Ax}\right)^2$. From which: $x^4 - 100x^2 + 204 = 0$.
(iv) The roots: $x_1^2 = 64$, and $x_2^2 = 36$. The corresponding values of $y$ are found from $y = \sqrt{x^2 - x'^2}$ from which $(x_1, y_1) = (8, 6)$, and $(x_2, y_2) = (6, 8)$.
(v) Substitute these pairs to obtain the possible values of the area moments of inertia:

$I_{x_1} = Ax_1^2 + I_{xc} = 4020$ mm$^4$,

$I_{x_2} = Ax_2^2 + I_{xc} = 6980$ mm$^4$,

$I_{y_1} = Ax_1^2 + I_{xy} = 6820$ mm$^4$,

$I_{y_2} = Ax_2^2 + I_{xy} = 4180$ mm$^4$
Problem 8.81 Determine the moment of inertia of the beam cross section about the $x$ axis. Compare your result with the moment of inertia of a solid square cross section of equal area. (See Example 8.5.)

Solution: We first need to find the location of the centroid of the composite. Break the area into two parts. Use $X$, $Y$ coords.

For the composite

$X_c = \frac{X_1A_1 + X_2A_2}{A_1 + A_2}$
$Y_c = \frac{Y_1A_1 + Y_2A_2}{A_1 + A_2}$

Substituting, we get

$X_c = 0$ mm
$Y_c = 114.6$ mm

We now find $I_x$ for each part about its center and use the parallel axis theorem to find $I_x$ about $C$.

For part (1): $b_1 = 100$ mm, $h_1 = 20$ mm

$I_{x1} = \frac{1}{12} b_1 h_1^3 = \frac{1}{12} (100)(20)^3 \text{ mm}^4$

$I_{x1} = 6.667 \times 10^4 \text{ mm}^4$

$dY_1 = Y_{c1} - Y_c = 55.38$ mm

$I_{x1} = I_{x1} + (dY_1)^2 A_1$

$I_{x1} = 6.20 \times 10^6 \text{ mm}^4$

For part (2): $b_2 = 20$ mm, $h_2 = 160$ mm

$I_{x2} = \frac{1}{12} (b_2)(h_2)^3 = \frac{1}{12} (20)(160)^3 \text{ mm}^4$

$I_{x2} = 6.827 \times 10^6 \text{ mm}^4$

$dY_2 = Y_{c2} - Y_c = -34.61$ mm

$I_{x2} = I_{x2} + (dY_2)^2 A_2$

$I_{x2} = 1.866 \times 10^7 \text{ mm}^4$

Finally, $I_x = I_{x1} + I_{x2}$

$I_x = 1.686 \times 10^7 \text{ mm}^4$

for our composite shape.

Now for the comparison. For the solid square with the same total area $A_1 + A_2 = 5200 \text{ mm}^2$, we get a side of length

$l^2 = 5200 \Rightarrow l = 72.11$ mm

And for this solid section

$I_{x,sq} = \frac{1}{12} bl^4 = \frac{1}{12} l^4$

$I_{x,sq} = 2.253 \times 10^6 \text{ mm}^4$

Ratio $= \frac{I_x}{I_{x,sq}} = \frac{1.686 \times 10^7}{2.253 \times 10^6}$

Ratio $= 7.48$

This matches the value in Example 8.5.
Problem 8.82  The area of the beam cross section is 5200 mm². Determine the moment of inertia of the beam cross section about the x axis. Compare your result with the moment of inertia of a solid square cross section of equal area. (See Example 8.5.)

Solution:  Let the outside dimension be \( b \) mm, then the inside dimension is \( b - 40 \) mm. The cross section is \( A = b^2 - (b - 40)^2 = 5200 \) mm². Solve: \( b = 85 \) mm. Divide the beam cross section into two parts: the inner and outer squares. Part (1)

\[
A_1 = 85^2 = 7225 \text{ mm}^2,
\]

\[
I_{xx1} = \left( \frac{1}{12} \right) 85(85^3) = 4.35 \times 10^6.
\]

Part (2)

\[
A_2 = 45^2 = 2025 \text{ mm}^2,
\]

\[
I_{xx2} = \left( \frac{1}{12} \right) 45(45^3) = 3.417 \times 10^5.
\]

The composite moment of inertia about the centroid is

\[
I_x = I_{xx1} - I_{xx2} = 4.008 \times 10^6 \text{ mm}^4.
\]

For a square cross section of the same area, \( h = \sqrt{5200} = 72.111 \) mm.

The area moment of inertia is

\[
I_{ab} = \left( \frac{1}{12} \right) 72.111(72.111^3) = 2.253 \times 10^6 \text{ in}^4.
\]

The ratio:

\[
R = \frac{4.008 \times 10^6}{2.253 \times 10^6} = 1.7788 = 1.78
\]

which confirms the value given in Example 8.5.
Problem 8.83 If the beam in Fig. a is subjected to couples of magnitude $M$ about the $x$ axis (Fig. b), the beam’s longitudinal axis bends into a circular arc whose radius $R$ is given by

$$R = \frac{EI_x}{M}$$

where $I_x$ is the moment of inertia of the beam’s cross section about the $x$ axis. The value of the term $E$, which is called the modulus of elasticity, depends on the material of which the beam is constructed. Suppose that a beam with the cross section shown in Fig. c is subjected to couples of magnitude $M = 180$ N-m. As a result, the beam’s axis bends into a circular arc with radius $R = 3$ m. What is the modulus of elasticity of the beam’s material? (See Example 8.5.)

Solution: The moment of inertia of the beam’s cross section about the $x$ axis is

$$I_x = \left\{ \frac{1}{12}(3)(9)^3 + 2\left[\frac{1}{12}(9)(3)^3 + (6)^2(9)(3)\right] \right\} \text{ mm}^4$$

$$= 2170 \text{ mm}^4 = 2.17 \times 10^{-9} \text{ m}^4.$$  

The modulus of elasticity is

$$E = \frac{RM}{I_x} = \frac{(3 \text{ m})(180 \text{ N-m})}{2.17 \times 10^{-9} \text{ m}^4} = 2.49 \times 10^{11} \text{ N/m}^2$$

$$E = 2.49 \times 10^{11} \text{ N/m}^2.$$
**Problem 8.84** Suppose that you want to design a beam made of material whose density is 8000 kg/m³. The beam is to be 4 m in length and have a mass of 320 kg. Design a cross section for the beam so that \( I_x = 5 \times 10^{-3} \) m⁴. (See Example 8.5.)

**Solution:** The strategy is to determine the cross sectional area, and then use the ratios given in Figure 8.14 to design a beam. The volume of the beam is \( V = AL = 4 A \) m³. The mass of the beam is \( m = V(8000) = 32000A = 320 \) kg, from which \( A = 0.01 \) m². The moment of inertia for a beam of square cross section with this area is

\[
I_{cam} = \left( \frac{1}{12} \right) (0.1)(0.1^2) = 8.333 \times 10^{-6} \text{ m}^4.
\]

The ratio is \( R = \frac{3 \times 10^{-5}}{8.333 \times 10^{-6}} = 3.6.\)

From Figure 8.6, this ratio suggests an I-beam of the form shown in the sketch. Choose an I-beam made up of three equal area rectangles, of dimensions \( b \) by \( hm \) in section. The moment of inertia about the centroid is \( I_x = y_1^2A_1 + I_{xx1} + y_2^2A_2 + I_{xx2} + y_3^2A_3 + I_{xx3}.\)

Since all areas are equal, \( A_1 = A_2 = A_3 = bh, \) and \( y_1 = \frac{b + h}{2}, y_2 = 0, \) and \( y_3 = -y_2, \) this reduces to

\[
I_x = \left( \frac{1}{6} \right) bh^3 + 2 \left( \frac{b + h}{2} \right)^2 bh + \left( \frac{1}{12} \right) bh^3.
\]

Note that \( bh = \frac{A}{1}, \) where \( A \) is the known total cross section area. These are two equations in two unknowns. Plot the function

\[
f(b) = \left( \frac{1}{6} \right) bh^3 + 2 \left( \frac{b + h}{2} \right)^2 bh + \left( \frac{1}{12} \right) bh^3 - I_x
\]

subject to the condition that \( bh = \frac{A}{1}. \) The function was graphed using **TK Solver Plus.** The graph crosses the zero axis at approximately \( b = 0.0395 \) m and \( b = 0.09 \) m. The lower value is an allowable value for \( b \) and the greater value corresponds to an allowable value of \( h. \) Thus the I beam design has the flange dimensions, \( b = 90 \) mm and \( h = 39.5 \) mm.

---

**Problem 8.85** The area in Fig. (a) is a C230×30 American Standard Channel beam cross section. Its cross sectional area is \( A = 3790 \) mm² and its moments of inertia about the \( x \) and \( y \) axes are \( I_x = 25.3 \times 10^6 \) mm⁴ and \( I_y = 1 \times 10^8 \) mm⁴. Suppose that two beams with C230×30 cross sections are riveted together to obtain a composite beam with the cross section shown in Fig. (b). What are the moments of inertia about the \( x \) and \( y \) axes of the composite beam?

**Solution:**

\[
I_x = 2(25.3 \times 10^6 \text{ mm}^4) = 50.6 \times 10^6 \text{ mm}^4
\]

\[
I_y = 2(10^8 \text{ mm}^4 + [3790 \text{ mm}^2][14.8 \text{ mm}]^2) = 3.66 \times 10^8 \text{ mm}^4
\]

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**Problem 8.86** The area in Fig. (a) is an L152×102×12.7 Angle beam cross section. Its cross sectional area is \( A = 3060 \text{ mm}^2 \) and its moments of inertia about the \( x \) and \( y \) axes are \( I_x = 7.24 \times 10^6 \text{ mm}^4 \) and \( I_y = 2.61 \times 10^6 \text{ mm}^4 \). Suppose that four beams with L152×102×12.7 cross sections are riveted together to obtain a composite beam with the cross section shown in Fig. (b). What are the moments of inertia about the \( x \) and \( y \) axes of the composite beam?

**Solution:**

\[
\begin{align*}
I_x &= 4(7.24 \times 10^6 \text{ mm}^4 + [3060 \text{ mm}^2][50.2 \text{ mm}]) \\
&= 59.8 \times 10^6 \text{ mm}^4 \\
I_y &= 4(2.61 \times 10^6 \text{ mm}^4 + [3060 \text{ mm}^2][24.9 \text{ mm}]) \\
&= 18.0 \times 10^6 \text{ mm}^4
\end{align*}
\]

**Problem 8.87** In Active Example 8.6, suppose that the vertical 3-m dimension of the triangular area is increased to 4 m. Determine a set of principal axes and the corresponding principal moments of inertia.

**Solution:** From Appendix B, the moments and products of inertia of the area are

\[
\begin{align*}
I_x &= \frac{1}{12}(4 \text{ m})(4 \text{ m})^3 = 21.3 \text{ m}^4, \\
I_y &= \frac{1}{4}(4 \text{ m})^3(4 \text{ m}) = 64 \text{ m}^4, \\
I_{xy} &= \frac{1}{8}(4 \text{ m})^2(4 \text{ m})^2 = 32 \text{ m}^4.
\end{align*}
\]

From Eq. (8.26),

\[
\tan(2\theta_p) = \frac{2I_{xy}}{I_y-I_x} = \frac{2(32)}{64-21.3} = 1.50 \Rightarrow \theta_p = 28.2^\circ.
\]

From Eqs. (8.23) and (8.24), the principal moments of inertia are

\[
\begin{align*}
I_x &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta_p - I_{xy} \sin 2\theta_p \\
&= \left( \frac{21.3+64}{2} \right) + \left( \frac{21.3-64}{2} \right) \cos(2(28.2^\circ)) - (32) \sin(2(28.2^\circ)) \\
&= 4.21 \text{ m}^4 \\
I_y &= \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta_p + I_{xy} \sin 2\theta_p \\
&= \left( \frac{21.3+64}{2} \right) - \left( \frac{21.3-64}{2} \right) \cos(2(28.2^\circ)) + (32) \sin(2(28.2^\circ)) \\
&= 81.8 \text{ m}^4
\end{align*}
\]

\[\theta_p = 28.2^\circ, \text{ principal moments of inertia are } 4.21 \text{ m}^4, 81.8 \text{ m}^4\]
Problem 8.88 In Example 8.7, suppose that the area is reoriented as shown. Determine the moments of inertia \( I_x', I_y' \) and \( I_{xy}' \) if \( \theta = 30^\circ \).

Solution: Based on Example 8.7, the moments and product of inertia of the reoriented area are

\[
I_x = 10 \text{ m}^4, \quad I_y = 22 \text{ m}^4, \quad I_{xy} = 6 \text{ m}^4.
\]

Applying Eqs. (8.23)–(8.25),

\[
I_x' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta
\]

\[
= \frac{10 + 22}{2} + \frac{10 - 22}{2} \cos 60^\circ - 6 \sin 60^\circ = 7.80 \text{ m}^4,
\]

\[
I_y' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta
\]

\[
= \frac{10 + 22}{2} + \frac{10 - 22}{2} \cos 60^\circ + 6 \sin 60^\circ = 24.2 \text{ m}^4,
\]

\[
I_{xy}' = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta
\]

\[
= \frac{12 - 22}{2} \sin 60^\circ + 6 \cos 60^\circ = -2.20 \text{ m}^4.
\]

\[
I_x = 7.80 \text{ m}^4, I_y = 24.2 \text{ m}^4, I_{xy}' = -2.20 \text{ m}^4.
\]

Problem 8.89 In Example 8.7, suppose that the area is reoriented as shown. Determine a set of principal axes and the corresponding principal moments of inertia. Based on the result of Example 8.7, can you predict a value of \( \theta_p \) without using Eq. (8.26)?

Solution: Based on Example 8.7, the moments and product of inertia of the reoriented area are

\[
I_x = 10 \text{ m}^4, \quad I_y = 22 \text{ m}^4, \quad I_{xy} = 6 \text{ m}^4.
\]

From Eq. (8.26),

\[
\tan 2\theta_p = \frac{2I_{xy}}{I_x - I_y} = \frac{2 \times 6}{22 - 10} = 1 \Rightarrow \theta_p = 22.5^\circ
\]

This value could have been anticipated from Example 8.7 by reorienting the axes.

Substituting the angle into Eqs. (8.23) and (8.24), the principal moments of inertia are

\[
I_x' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta_p - I_{xy} \sin 2\theta_p
\]

\[
= \frac{10 + 22}{2} + \frac{10 - 22}{2} \cos 45^\circ - 6 \sin 45^\circ = 7.51 \text{ m}^4,
\]

\[
I_y' = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta_p + I_{xy} \sin 2\theta_p
\]

\[
= \frac{10 + 22}{2} - \frac{10 - 22}{2} \cos 45^\circ + 6 \sin 45^\circ = 24.5 \text{ m}^4,
\]

\[
\theta_p = 22.5^\circ, \text{ principal moments of inertia are 7.51 m}^4, 24.5 m^4.
\]
Problem 8.90  The moment of inertia of the area are

\[ I_x = 1.26 \times 10^6 \text{ cm}^4, \]
\[ I_y = 6.55 \times 10^5 \text{ cm}^4, \]
\[ I_{xy} = -1.02 \times 10^5 \text{ cm}^4 \]

Determine the moments of inertia of the area \( I'_{x}, I'_{y}, \) and \( I'_{x'y'} \) if \( \theta = 30^\circ \).

Solution:
Applying Eqs. (8.23)–(8.25),

\[ I'_x = \frac{I_x + I_y}{2} + \frac{I_y - I_x}{2} \cos 2\theta - I_{xy} \sin 2\theta \]
\[ = \left[ \frac{1.26 + 6.55}{2} + \frac{6.55 - 1.26}{2} \cos 60^\circ - (-0.102) \sin 60^\circ \right] \times 10^6 \text{ cm}^4 \]
\[ = 1.20 \times 10^6 \text{ cm}^4 \]

\[ I'_y = \frac{I_x + I_y}{2} - \frac{I_y - I_x}{2} \cos 2\theta + I_{xy} \sin 2\theta \]
\[ = \left[ \frac{1.26 + 6.55}{2} - \frac{6.55 - 1.26}{2} \cos 60^\circ + (-0.102) \sin 60^\circ \right] \times 10^6 \text{ cm}^4 \]
\[ = 7.18 \times 10^5 \text{ cm}^4 \]

\[ I'_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \]
\[ = \left[ \frac{1.26 - 6.55}{2} \sin 60^\circ + (-0.102) \cos 60^\circ \right] \times 10^6 \text{ cm}^4 \]
\[ = 2.11 \times 10^5 \text{ cm}^4 \]

\[ I'_x = 1.20 \times 10^6 \text{ cm}^4, I'_y = 7.18 \times 10^5 \text{ cm}^4, I'_{x'y'} = 2.11 \times 10^5 \text{ cm}^4 \]
Problem 8.91  The moment of inertia of the area are

\[ I_x = 1.26 \times 10^6 \text{ cm}^4, \]
\[ I_y = 6.55 \times 10^5 \text{ cm}^4, \]
\[ I_{xy} = -1.02 \times 10^5 \text{ cm}^4 \]

Determine a set of principal axes and the corresponding principal moments of inertia.

Solution: From Eq. (8.26),

\[ \tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{2(-102)}{0.655 - 1.26} = 0.337 \]

\[ \Rightarrow \theta = 9.32^\circ \]

Substituting this angle into Eqs. (8.23) and (8.24), the principal moments of inertia are

\[ I_x' = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \]
\[ = \left\{ \frac{1.26 + 0.655}{2} + \frac{1.26 - 0.655}{2} \cos 18.63^\circ - (-0.102) \sin 18.63^\circ \right\} \times 10^6 \text{ cm}^4 = 1.28 \times 10^6 \text{ cm}^4 \]
\[ I_y' = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \]
\[ = \left\{ \frac{1.26 + 0.655}{2} - \frac{1.26 - 0.655}{2} \cos 18.63^\circ + (-0.102) \sin 18.63^\circ \right\} \times 10^6 \text{ cm}^4 = 6.38 \times 10^5 \text{ cm}^4 \]

\[ \theta = 9.32^\circ, \text{ principal moments of inertia are} \]
\[ 1.28 \times 10^6 \text{ cm}^4, 6.38 \times 10^5 \text{ cm}^4. \]
Problem 8.92* Determine a set of principal axes and the corresponding principal moments of inertia.

Solution: We divide the area into 3 rectangles as shown: In terms of the \( \hat{x}, \hat{y} \) coordinate system, the position of the centroid is

\[
\bar{x} = \frac{\hat{x}_1 A_1 + \hat{x}_2 A_2 + \hat{x}_3 A_3}{A_1 + A_2 + A_3} = \frac{(20)(40)(200) + (100)(120)(40) + (80)(80)(40)}{(40)(200) + (120)(40) + (80)(40)} \approx 56 \text{ mm},
\]

\[
\bar{y} = \frac{\hat{y}_1 A_1 + \hat{y}_2 A_2 + \hat{y}_3 A_3}{A_1 + A_2 + A_3} = \frac{(100)(40)(200) + (180)(120)(40) + (20)(80)(40)}{(40)(200) + (120)(40) + (80)(40)} \approx 108 \text{ mm}.
\]

The moments and products of inertia in terms of the \( \hat{x}, \hat{y} \) system are

\[
\hat{I}_x = (\hat{I}_{x1})_1 + (\hat{I}_{x2})_2 + (\hat{I}_{x3})_3
= \frac{1}{3}(40)(200)^3 + \frac{1}{12}(120)(40)^3 + (180)^2(120)(40)
+ \frac{1}{3}(80)(40)^3 = 26.5 \times 10^7 \text{ mm}^4,
\]

\[
\hat{I}_y = (\hat{I}_{y1})_1 + (\hat{I}_{y2})_2 + (\hat{I}_{y3})_3
= \frac{1}{3}(200)(40)^3 + \frac{1}{12}(140)(120)^3 + (100)^2(120)(40)
+ \frac{1}{12}(40)(80)^3 + (80)^2(80)(40) = 8.02 \times 10^7 \text{ mm}^4,
\]

\[
\hat{I}_{xy} = (\hat{I}_{xy1})_1 + (\hat{I}_{xy2})_2 + (\hat{I}_{xy3})_3
= (20)(100)(40)(200) + (100)(180)(40)(120)
+ (20)(80)(40)(80) = 10.75 \times 10^7 \text{ mm}.
\]

The moments and product of inertia in terms of the \( \hat{x}, \hat{y} \) system are

\[
I_x = \hat{I}_x - (\bar{y})^2 A = 77.91 \times 10^6 \text{ mm}^4,
\]

\[
I_y = \hat{I}_y - (\bar{x})^2 A = 30.04 \times 10^6 \text{ mm}^4,
\]

\[
I_{xy} = \hat{I}_{xy} - \bar{x}\bar{y} A = 10.75 \times 10^6 \text{ mm}^4.
\]

from Equation (8.26),

\[
\tan 2\theta_p = \frac{2I_{xy}}{I_y - I_x} = \frac{2(10.75 \times 10^6)}{(30.04 \times 10^6) - (77.91 \times 10^6)}.
\]

we obtain \( \theta_p = -12.1^\circ \). We can orient the principal axes as shown: Substituting the values of \( I_x, I_y \) and \( I_{xy} \) into Equations (8.23) and (8.24) and setting \( \theta = -12.1^\circ \), we obtain

\[
I_{ix} = 80.2 \times 10^6 \text{ mm}^4
\]

\[
I_{iy} = 27.7 \times 10^6 \text{ mm}^4.
\]

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**Problem 8.93** Solve Problem 8.87 by using Mohr’s Circle.

**Solution:** The vertical 3-m dimension is increased to 4 m. From Problem 8.87, the moments and product of inertia for the unrotated system are

\[ I_x = \frac{1}{12}(4 \text{ m})(4 \text{ m})^3 = 21.3 \text{ m}^4, \]

\[ I_y = \frac{1}{4}(4 \text{ m})^2(4 \text{ m}) = 64 \text{ m}^4, \]

\[ I_{xy} = \frac{1}{8}(4 \text{ m})^2(4 \text{ m})^2 = 32 \text{ m}^4. \]

Mohr’s circle (shown) has a center and radius given by

\[ C = \frac{21.3 + 64}{2} = 42.7 \text{ m}^4 \]

\[ R = \sqrt{\left(\frac{21.3 - 64}{2}\right)^2 + (32)^2} = 38.5 \text{ m}^4 \]

The angle and principal moments are now

\[ \tan(2\theta_p) = \frac{32}{64 - 42.7} \Rightarrow \theta_p = 28.2^\circ, \]

\[ I_1 = C + R = 81.1 \text{ m}^4, I_2 = C - R = 4.21 \text{ m}^4. \]

\[ \theta_p = 28.2^\circ, \text{ principal moments of inertia are } 4.21 \text{ m}^4, 81.1 \text{ m}^4. \]
Problem 8.94  Solve Problem 8.88 by using Mohr’s Circle.

Solution: Based on Example 8.7, the moments and product of inertia of the reoriented area are

\[ I_x = 10 \text{ m}^4, \quad I_y = 22 \text{ m}^4, \quad I_{xy} = 6 \text{ m}^4. \]

For Mohr’s circle we have the center, radius, and angle

\[ C = \frac{10 + 22}{2} = 16 \text{ m}^4, \]
\[ R = \sqrt{\left(\frac{22 - 10}{2}\right)^2 + (6)^2} = 8.49 \text{ m}^4, \]
\[ \theta_R = \frac{1}{2} \tan^{-1}\left(\frac{6}{22 - 10}\right) = 22.5^\circ. \]

Now we can calculate the new inertias

\[ I_x' = C - R \cos 15^\circ = 7.80 \text{ m}^4 \]
\[ I_y' = C + R \cos 15^\circ = 24.2 \text{ m}^4 \]
\[ I_{xy}' = -R \sin 15^\circ = -2.20 \text{ m}^4 \]

\[ I_x' = 7.80 \text{ m}^4, \quad I_y' = 24.2 \text{ m}^4, \quad I_{xy}' = -2.20 \text{ m}^4. \]
**Problem 8.95**  Solve Problem 8.89 by using Mohr’s Circle.

**Solution:** Based on Example 8.7, the moments and product of inertia of the reoriented area are

\[ I_x = 10 \text{ m}^4, I_y = 22 \text{ m}^4, I_{xy} = 6 \text{ m}^4. \]

For Mohr’s circle we have the center, radius, and angle

\[ C = \frac{10 + 22}{2} = 16 \text{ m}^4, \]
\[ R = \sqrt{\left( \frac{22 - 10}{2} \right)^2 + (6)^2} = 8.49 \text{ m}^4, \]
\[ \theta_p = \frac{1}{2} \tan^{-1} \left( \frac{6}{22 - 16} \right) = 22.5^\circ. \]

Now we can calculate the principal moments of inertias

\[ I_1 = C + R = 24.5 \text{ m}^4 \]
\[ I_2 = C - R = 7.51 \text{ m}^4 \]

\[ \theta_p = 22.5^\circ; \text{ principal moments of inertia are } 7.51 \text{ m}^4, 24.5 \text{ m}^4. \]
Problem 8.96  Solve Problem 8.90 by using Mohr's Circle.

Solution:  For Mohr's circle we have the center, radius, and angle 
\[ C = \left( \frac{12.6 + 6.55}{2} \right) \times 10^3 = 9.58 \times 10^3 \text{ in}^4. \]
\[ R = \sqrt{\left( \frac{12.6 - 6.55}{2} \right)^2 + (-1.02)^2} \times 10^3 = 3.19 \text{ in}^4. \]
\[ \theta_p = \frac{1}{2} \tan^{-1} \left( \frac{1.02}{12.6 - 9.58} \right) = 9.32^\circ \]

Now we can calculate the new inertias
\[ I_{x'} = C + R \cos 22.7^\circ = 12.0 \times 10^3 \text{ in}^4, \]
\[ I_{y'} = C - R \cos 22.7^\circ = 7.18 \times 10^3 \text{ in}^4, \]
\[ I_{xy'} = R \sin 22.7^\circ = 2.11 \times 10^3 \text{ in}^4. \]

\[ I_x = 1.20 \times 10^6 \text{ in}^4, I_y = 7.18 \times 10^5 \text{ in}^4, I_{xy} = 2.11 \times 10^5 \text{ in}^4. \]
Problem 8.97  Solve Problem 8.91 by using Mohr’s Circle.

Solution:  For Mohr’s circle we have the center, radius, and angle
\[ C = \left( \frac{12.6 + 6.55}{2} \right) \times 10^5 = 9.58 \times 10^5 \text{ in}^4, \]
\[ R = \sqrt{\left( \frac{12.6 - 6.55}{2} \right)^2 + (-1.02)^2} \times 10^5 = 3.19 \text{ in}^4, \]
\[ \theta_p = \frac{1}{2} \tan^{-1} \left( \frac{1.02}{12.6 - 9.38} \right) = 9.32^\circ. \]

Now we can calculate the principal inertias
\[ I_1 = C + R = 12.8 \times 10^5 \text{ in}^4, \]
\[ I_{y'} = C - R = 6.38 \times 10^5 \text{ in}^4. \]

\[ \theta_p = 9.32^\circ, \] principal moments of inertia are
\[ 1.28 \times 10^6 \text{ in}^4, 6.38 \times 10^5 \text{ in}^4. \]
Problem 8.98*  Solve Problem 8.92 by using Mohr's circle.

Solution:  The moments and product of inertia are derived in terms of the $x'y$ coordinate system in the solution of Problem 8.92:

\[
\begin{align*}
I_x &= 77.91 \times 10^6 \text{ mm}^4 \\
I_y &= 30.04 \times 10^6 \text{ mm}^4 \\
I_{xy} &= 10.75 \times 10^6 \text{ mm}^4.
\end{align*}
\]

The Mohr's circle is: Measuring the $2\theta_p$, angle we estimate that $\theta_p = -12^\circ$, and the principle moments of inertia are approximately $81 \times 10^6 \text{ mm}^4$ and $28 \times 10^6 \text{ mm}^4$ the orientation of the principal axes is shown in the solution of Problem 8.92.

Problem 8.99  Derive Eq. (8.22) for the product of inertia by using the same procedure we used to derive Eqs. (8.20) and (8.21).

Solution:  Suppose that the area moments of inertia of the area $A$ are known in the coordinate system $(x, y)$,

\[
\begin{align*}
I_x &= \int_A y^2 \, dA, \\
I_y &= \int_A x^2 \, dA, \\
I_{xy} &= \int_A xy \, dA.
\end{align*}
\]

The objective is to find the product of inertia in the new coordinate system $(x', y')$ in terms of the known moments of inertia. The new $(x', y')$ system is formed from the old $(x, y)$ system by rotation about the origin through a counterclockwise angle $\theta$.

By definition,

\[
I_{x'y'} = \int_A x'y' \, dA.
\]

From geometry,

\[
x' = x \cos \theta + y \sin \theta,
\]

and

\[
y' = -x \sin \theta + y \cos \theta.
\]

The product is

\[
x'y' = xy \cos^2 \theta - xy \sin^2 \theta + y^2 \cos \theta \sin \theta - x^2 \cos \theta \sin \theta.
\]

Substitute into the definition:

\[
I_{x'y'} = (\cos^2 \theta - \sin^2 \theta) \int_A xy \, dA + (\cos \theta \sin \theta) \left( \int_A y^2 \, dA - \int_A x^2 \, dA \right),
\]

from which

\[
I_{x'y'} = (\cos^2 \theta - \sin^2 \theta) I_{xy} + (I_x - I_y) \sin \theta \cos \theta,
\]

which is the expression required.
Problem 8.100  The axis $L_0$ is perpendicular to both segments of the $L$-shaped slender bar. The mass of the bar is 6 kg and the material is homogeneous. Use the method described in Example 8.10 to determine its moment of inertia about $L_0$.

Solution:  Use Example 8.10 as a model for this solution.

Introduce the coordinate system shown and divide the bar into two parts as shown:

\[
\begin{align*}
(I_0)_1 &= \int r^2 \, dm = \int_0^1 \rho A x^2 \, dx = \rho A \left[ \frac{x^3}{3} \right]_0^1 = \frac{\rho A}{3} \\
(I_0)_2 &= \int_1^2 r^2 \, dm = \rho A \int_1^2 (2^2 + y^2) \, dy
\end{align*}
\]

However $m_1 = \rho A l_1 = (2/3)\rho A$.

Since part 1 is 2/3 of the length, its mass is 2/3(6 kg) = 4 kg. Part 2 has mass 2 kg.

For part 2, $dm = \rho A \, dy$ and

\[
r = \sqrt{2^2 + y^2}
\]

\[
(I_0)_2 = \rho A \int_1^2 (2^2 + y^2) \, dy = \rho A \left[ \frac{2y^2}{2} + \frac{y^3}{3} \right]_1^2 = \rho A \left[ \frac{13}{3} \right]
\]

\[
I_{0\text{TOTAL}} = \frac{13}{3} \rho A + \frac{8}{3} \rho A = \frac{21}{3} \rho A
\]

\[
(I_0)_{\text{TOTAL}} = 7\rho A
\]

The total mass is $3\rho A = 6$ kg.

\[
I_{0\text{TOTAL}} = \frac{7}{6}(6 \text{ kg}) = 7 \text{ kg m}^2
\]
Problem 8.101  Two homogenous slender bars, each of mass \( m \) and length \( l \), are welded together to form the T-shaped object. Use integration to determine the moment of inertia of the object about the axis through point \( O \) that is perpendicular to the bars.

**Solution:** Divide the object into two pieces, each corresponding to a slender bar of mass \( m \); the first parallel to the \( y \) axis, the second to the \( x \) axis. By definition

\[
I = \int_0^l r^2 \, dm + \int_0^l r^2 \, dm.
\]

For the first bar, the differential mass is \( dm = \rho A \, dx \). Assume that the second bar is very slender, so that the mass is concentrated at a distance \( l \) from \( O \). Thus \( dm = \rho A \, dx \), where \( x \) lies between the limits

\[
\frac{l}{2} \leq x \leq \frac{l}{2}.
\]

The distance to a differential \( dx \) is \( r = \sqrt{l^2 + x^2} \). Thus the definition becomes

\[
I = \rho A \left( \int_0^l x^2 \, dx + \int_0^l (l^2 + x^2) \, dx \right)
= \rho A \left( \frac{1}{3} l^3 + \frac{1}{5} l^5 \right)
= ml^2 \left( \frac{1}{3} + \frac{1}{5} \right) = \frac{17}{15} ml^2
\]

Problem 8.102  The slender bar lies in the \( x – y \) plane. Its mass is 6 kg and the material is homogeneous. Use integration to determine its moment of inertia about the \( z \) axis.

**Solution:** The density is \( \rho = \frac{6 \text{ kg}}{3 \text{ m}} = 2 \text{ kg/m} \)

\[
I_z = \int_0^1 \rho x^2 \, dx + \int_0^2 \rho \left(1 + \frac{x}{\cos 50^\circ} \right)^2 \, dx
\]

\[
I_z = 15.14 \text{ kg m}^2
\]

Problem 8.103  Use integration to determine the moment of inertia of the slender bar in Problem 8.102 about the \( y \) axis.

**Solution:** See solution for 8.102

\[
I_y = \int_0^1 \rho x^2 \, dx + \int_0^2 \rho (1 + \frac{x}{\cos 50^\circ})^2 \, dx = 12.01 \text{ kg m}^2
\]
Problem 8.104  The homogeneous thin plate has mass \( m = 12 \) kg and dimensions \( b = 2 \) m and \( h = 1 \) m. Use the procedure described in Active Example 8.9 to determine the moments of inertia of the plate about the \( x \) and \( y \) axes.

**Solution:** From Appendix B, the moments of inertia about the \( x \) and \( y \) axes are

\[
I_x = \frac{1}{12} bh^2, \quad I_y = \frac{1}{12} hb^2.
\]

Therefore the moments of inertia of the plate about the \( x \) and \( y \) axes are

\[
I_{x,\text{axis}} = \frac{m}{A} I_x = \frac{m}{A} \left( \frac{1}{12} bh^2 \right) = \frac{1}{18} (12 \text{ kg})(1 \text{ m})^2 = 0.667 \text{ kg-m}^2
\]

\[
I_{y,\text{axis}} = \frac{m}{A} I_y = \frac{m}{A} \left( \frac{1}{12} hb^2 \right) = \frac{1}{18} (12 \text{ kg})(2 \text{ m})^2 = 2.67 \text{ kg-m}^2
\]

\[
I_{x,\text{axis}} = 0.667 \text{ kg-m}^2; \quad I_{y,\text{axis}} = 2.67 \text{ kg-m}^2.
\]

Problem 8.105  The homogeneous thin plate is of uniform thickness and mass \( m \).

(a) Determine its moments of inertia about the \( x \) and \( z \) axes.

(b) Let \( R_i = 0 \), and compare your results with the values given in Appendix C for a thin circular plate.

**Solution:**

(a) The area moments of inertia for a circular area are \( I_x = I_y = \frac{\pi R^4}{4} \). For the plate with a circular cutout, \( I_x = \frac{\pi}{4}(R_o^4 - R_i^4) \). The area mass density is \( \frac{m}{A} \), thus for the plate with a circular cutout, \( \frac{m}{A} = \frac{m}{\pi(R_o^2 - R_i^2)} \), from which the moments of inertia

\[
I_{x,\text{axis}} = \frac{m(R_o^4 - R_i^4)}{4(R_o^2 - R_i^2)} = \frac{m}{4}(R_o^2 + R_i^2)
\]

\[
I_{z,\text{axis}} = 2I_{x,\text{axis}} = \frac{m}{2}(R_o^2 + R_i^2).
\]

(b) Let \( R_i = 0 \), to obtain

\[
I_{x,\text{axis}} = \frac{m}{4} R_o^2
\]

\[
I_{z,\text{axis}} = \frac{m}{2} R_o^2
\]

which agrees with table entries.
Problem 8.106  The homogenous thin plate is of uniform thickness and weighs 200 N. Determine its moment of inertia about the y axis.

Solution:

\[ y = 4 - \frac{1}{4} x^2 \text{ m} \]

The plate's area is

\[ A = \int_{-4}^{4} \left( 4 - \frac{1}{4} x^2 \right) \, dx = 21.3 \text{ m}^2. \]

The plate's density per unit area is

\[ \delta = \frac{200}{9.81} \text{ kg/m}^2. \]

The moment of inertia about the y axis is

\[ I_y = \int_{-4}^{4} x^2 \left( 4 - \frac{1}{4} x^2 \right) \, dx = 65.3 \text{ kg-m}^2. \]

Problem 8.107  Determine the moment of inertia of the plate in Problem 8.106 about the x axis.

Solution:  See the solution of Problem 8.106. The mass of the strip element is

\[ m_{strip} = \delta \left( 4 - \frac{1}{4} x^2 \right) \, dx. \]

The moment of inertia of the strip about the x axis is

\[ I_{x(strip)} = \frac{1}{3} m_{strip} \left( 4 - \frac{1}{4} x^2 \right)^2 \]

so the moment of inertia of the plate about the x axis is

\[ I_{x(plate)} = \int_{-4}^{4} \frac{1}{3} \left( 4 - \frac{1}{4} x^2 \right)^3 \, dx = 74.6 \text{ kg-m}^2. \]

Problem 8.108  The mass of the object is 10 kg. Its moment of inertia about \( L_1 \) is 10 kg-m\(^2\). What is its moment of inertia about \( L_2 \)? (The three axes lie in the same plane.)

Solution:  The strategy is to use the data to find the moment of inertia about \( L_1 \), from which the moment of inertia about \( L_2 \) can be determined.

\[ I_L = -(0.6)^2(10) + 10 = 6.4 \text{ m}^2, \]

from which \[ I_{L_2} = (1.2)^2(10) + 6.4 = 20.8 \text{ m}^2 \]
**Problem 8.109** An engineer gathering data for the design of a maneuvering unit determines that the astronaut’s center of mass is at $x = 1.01$ m, $y = 0.16$ m and that her moment of inertia about the $z$ axis is 105.6 kg-m². Her mass is 81.6 kg. What is her moment of inertia about the $z'$ axis through her center of mass?

**Solution:** The distance $d$ from the $z$ axis to the $z'$ axis is 

\[ d = \sqrt{(1.01)^2 + (0.16)^2} \]

\[ = 1.0226 \text{ m}. \]

From the parallel-axis theorem,

\[ I_{z \text{ axis}} = I_{z' \text{ axis}} + dm \]

\[ 105.6 = I_{z' \text{ axis}} + (1.0226)(81.6). \]

Solving, we obtain

\[ I_{z' \text{ axis}} = 20.27 \text{ kg-m}^2. \]

**Problem 8.110** Two homogenous slender bars, each of mass $m$ and length $l$, are welded together to form the T-shaped object. Use the parallel axis theorem to determine the moment of inertia of the object about the axis through point $O$ that is perpendicular to the bars.

**Solution:** Divide the object into two pieces, each corresponding to a bar of mass $m$. By definition

\[ I = \int_0^l r^2 \, dm. \]

For the first bar, the differential mass is $dm = \rho A \, dx$, from which the moment of inertia about one end is

\[ I_1 = \rho A \int_0^l r^2 \, dx = \rho A \left[ \frac{r^3}{3} \right]_0^l = \frac{ml^2}{3}. \]

For the second bar

\[ I_2 = \rho A \int_\frac{l}{2}^l r^2 \, dx = \rho A \left[ \frac{r^3}{3} \right]_\frac{l}{2}^l = \frac{ml^2}{12} \]

is the moment of inertia about the center of the bar. From the parallel axis theorem, the moment of inertia about $O$ is

\[ I_o = \frac{ml^2}{3} + l^2 m + \frac{ml^2}{12} = \frac{17}{12} ml^2. \]
**Problem 8.111** Use the parallel-axis theorem to determine the moment of inertia of the T-shaped object in Problem 8.110 about the axis through the center of mass of the object that is perpendicular to the two bars. (See Active Example 8.11.)

**Solution:** The location of the center of mass of the object is \( x = \frac{m\left(\frac{1}{4}\right) + lm}{2m} = \frac{1}{4}\). Use the results of Problem 8.110 for the moment of inertia of a bar about its center. For the first bar,

\[
I_1 = \left(\frac{1}{4}\right)^2 m + \frac{m l^2}{12} = \frac{7}{48} m^2.
\]

For the second bar,

\[
I_2 = \left(\frac{1}{4}\right)^2 m + \frac{m l^2}{12} = \frac{7}{48} m^2.
\]

The composite:

\[
I_c = I_1 + I_2 = \frac{7}{24} m^2.
\]

**Problem 8.112** The mass of the homogenous slender bar is 20 kg. Determine its moment of inertia about the \( z \) axis.

**Solution:** Divide the object into three segments. Part (1) is the 1 m bar on the left, Part (2) is the 1.5 m horizontal segment, and Part (3) is the segment on the far right. The mass density per unit length is

\[
\rho = \frac{m}{L} = \frac{20}{(1 + 1.5 + \sqrt{2})} = 5.11 \text{ kg/m}.
\]

The moments of inertia about the centers of mass and the distances to the centers of mass from the \( z \) axis are:

Part (1) \( I_1 = \rho \left(\frac{1}{12}\right) = m_1 \frac{l_1^2}{12} = 0.426 \text{ kg-m}^2 \).

\( m_1 = 5.11 \text{ kg} \),
\( d_1 = 0.5 \text{ m} \),

Part (2) \( I_2 = \rho \left(\frac{1}{12}\right) = m_2 \frac{l_2^2}{12} = 1.437 \text{ kg-m}^2 \).

\( m_2 = 7.66 \text{ kg} \),
\( d_2 = \sqrt{0.75^2 + l^2} = 1.25 \text{ m} \)

Part (3) \( I_3 = \rho \left(\frac{1}{12}\right) = m_3 \left(\frac{\sqrt{\pi^2}}{12}\right) = 1.204 \text{ kg-m}^2 \).

\( m_3 = 7.23 \text{ kg} \),
\( d_3 = \sqrt{2^2 + 0.5^2} = 2.062 \text{ m} \).

The composite:

\[
I = d_1^2 m_1 + I_1 + d_2^2 m_2 + I_2 + d_3^2 m_3 + I_3 = 47.02 \text{ kg-m}^2.
\]
**Problem 8.113** Determine the moment of inertia of the bar in Problem 8.112 about the \( z' \) axis through its center of mass.

**Solution:** The center of mass:

\[
x = \frac{\sum m_i x_i}{\sum m_i} = 0 + 0.75(7.66) + 2(7.23) = 1.01 \text{ m.}
\]

\[
y = \frac{\sum m_i y_i}{\sum m_i} = 0.5m_1 + 1m_2 + 0.5m_3 = 0.692 \text{ m.}
\]

The distance from the \( z \) axis to the center of mass is \( d = \sqrt{x^2 + y^2} = 1.224 \text{ m.} \) The moment of inertia about the center of mass:

\[
I_c = -d^2(20) + I_o
\]

\[
= 17.1 \text{ kg-m}^2
\]

**Problem 8.114** The homogeneous slender bar weighs 5 N. Determine its moment of inertia about the \( z \) axis.

**Solution:** The bar’s mass is \( m = 5/9.81 \text{ kg.} \) Its length is

\[
L = L_1 + L_2 + L_3 = 8 + \sqrt{8^2 + 8^2 + \pi(4)} = 31.9 \text{ cm.}
\]

The masses of the parts are therefore,

\[
m_1 = \frac{L_1}{L} m = \left( \frac{8}{31.9} \right) \left( \frac{5}{9.81} \right) = 0.1279 \text{ kg.}
\]

\[
m_2 = \frac{L_2}{L} m = \left( \frac{\sqrt{8^2 + 8^2 + \pi(4)}}{31.9} \right) \left( \frac{5}{9.81} \right) = 0.1809 \text{ kg.}
\]

\[
m_3 = \frac{L_3}{L} m = \left( \frac{4\pi}{31.9} \right) \left( \frac{5}{9.81} \right) = 0.2009 \text{ kg.}
\]

The center of mass of part 3 is located to the right of its center \( C' \) a distance \( 2R/\pi = 2(4)/\pi = 2.55 \text{ cm.} \) The moment of inertia of part 3 about \( C' \) is

\[
\int r^2 \, dm = m_3r^2 = (0.2009)(4)^2 = 3.2145 \text{ kg-cm}^2.
\]

The moment of inertia of part 3 about the center of mass of part 3 is therefore

\[
I_3 = 3.2145 - m_3(2.55)^2 = 1.9117 \text{ kg-cm}^2.
\]

The moment of inertia of the bar about the \( z \) axis is

\[
I_{(z \text{ axis})} = \frac{1}{3}m_1L_1^2 + \frac{1}{4}m_2L_2^2 + I_3 + m_3(8 + 2.55)^2 + (4)^2
\]

\[
= 37.9 \text{ kg-cm}^2 = 3.79 \times 10^{-3} \text{ kg-m}^2.
\]
Problem 8.115  Determine the moment of inertia of the bar in Problem 8.114 about the $z^*$ axis through its center of mass.

Solution: In the solution of Problem 8.114, it is shown that the moment of inertia of the bar about the $z$ axis is $I_z = 37.9 \text{ kg-cm}^2$. The $x$ and $y$ coordinates of the center of mass coincide with the centroid of the axis:

$$x = \frac{x_L + x_L + x_L}{L_2 + L_2 + L_3}$$

$$y = \frac{y_L + y_L + y_L}{L_2 + L_2 + L_3}$$

The moment of inertia about the $x$ axis is

$$I_{x^*} = I_z - (x^2 + y^2) \left( \frac{5}{32.2} \right) = 11.3 \text{ kg-cm}^2.$$

Problem 8.116  The rocket is used for atmospheric research. Its weight and its moment of inertia about the $z$ axis through its center of mass (including its fuel) are 50 kN and 13,770 kg-m$^2$, respectively. The rocket’s fuel weighs 30 kN, its center of mass is located at $x = -0.9 \text{ m}$, $y = 0$, $z = 0$, and the moment of inertia of the fuel about the axis through the fuel’s center of mass is 2970 kg-m$^2$. When the fuel is exhausted, what is the rocket’s moment of inertia about the axis through its new center of mass parallel to $z$?

Solution: Denote the moment of inertia of the empty rocket as $I_E$ about a mass center at the origin ($x_E = 0$). Using the parallel axis theorem, the moment of inertia of the filled rocket is

$$I_E = I_E + x_E^2 m_E + I_F + x_F^2 m_F,$$

about a mass center at the origin ($x_E = 0$). Solve:

$$I_E = I_E - x_E^2 m_E + I_F - x_F^2 m_F.$$  

The objective is to determine values for the terms on the right from the data given. Since the filled rocket has a mass center at the origin, the mass center of the empty rocket is found from $0 = m_E x_E + m_F x_F$, from which

$$x_E = - \left( \frac{m_F}{m_E} \right) x_F.$$  

Using a value of $g = 9.81 \text{ m/s}^2$,

$$m_F = \frac{W_F}{g} = \frac{6000}{9.81} = 3058.1 \text{ kg},$$

$$m_E = \frac{(W_E - W_F)}{g} = \frac{50,000 - 30,000}{9.81} = 2038.7 \text{ kg}.$$  

From which

$$x_E = - \left( \frac{3058.1}{2038.7} \right) (-0.9) = 1.35 \text{ m}$$

is the new location of the center of mass. Substitute:

$$I_E = I_E - x_E^2 m_E - I_F - x_F^2 m_F$$

$$= 13770 - 3715.6 - 2970 - 2477.1$$

$$= 4607.3 \text{ kg-m}^2.$$
Problem 8.117  The mass of the homogenous thin plate is 36 kg. Determine its moment of inertia about the $x$ axis.

**Solution:** Divide the plate into two areas: the rectangle 0.4 m by 0.6 m on the left, and the rectangle 0.4 m by 0.3 m on the right. The mass density is $\rho = \frac{m}{A}$. The area is

$$A = (0.4)(0.6) + (0.4)(0.3) = 0.36 \text{ m}^2,$$

from which

$$\rho = \frac{36}{0.36} = 100 \text{ kg/m}^2.$$

The moment of inertia about the $x$ axis is

$$I_x = \rho \left( \frac{1}{4} \right) (0.4)(0.6)^3 + \rho \left( \frac{1}{4} \right) (0.3)(0.4)^3$$

$$+ (0.6)^2 \rho (0.3)(0.4) = 5.76 \text{ kg-m}^2,$$

from which

$$I_x = 3.24 + 5.76 = 9 \text{ kg-m}^2.$$
Problem 8.120  Determine the moment of inertia of the plate in Problem 8.119 about the $y$ axis.

**Solution:** Use the results of the solution in Problem 8.119 for the area and the mass density.

\[
I_y = \rho \left( \frac{1}{12} \right) 5(10^3) + \rho \left( \frac{5}{3} \right) 5(5^3)
\]

\[
+ \rho \left( 5 + \frac{10}{3} \right) \left( \frac{1}{2} \right) 5(5)
\]

\[
= 41.62 \text{ kg-cm}^2 = 4.162 \times 10^{-3} \text{ kg-m}^2
\]

Problem 8.121  The thermal radiator (used to eliminate excess heat from a satellite) can be modeled as a homogenous, thin rectangular plate. Its mass is 80 kg. Determine its moment of inertia about the $x$, $y$, and $z$ axes.

**Solution:** The area is $A = 2.7 \times 0.9 = 2.43 \text{ m}^2$. The mass density is

\[
\rho = \frac{m}{A} = \frac{80}{2.43} = 32.922 \text{ kg/m}^2.
\]

The moment of inertia about the centroid of the rectangle is

\[
I_x = \rho \left( \frac{1}{12} \right) 2.7(0.9^3) = 5.4 \text{ kg-m}^2.
\]

\[
I_y = \rho \left( \frac{1}{12} \right) 0.9(2.7^3) = 48.6 \text{ kg-m}^2.
\]

Use the parallel axis theorem:

\[
I_x = \mu^2(0.6 + 0.45)^2 + I_x = 93.6 \text{ kg-m}^2.
\]

\[
I_y = \mu^2(1.35 - 0.9)^2 + I_y = 64.8 \text{ kg-m}^2.
\]

\[
I_z = I_x + I_y = 158.4 \text{ kg-m}^2
\]
Problem 8.122  The homogeneous cylinder has mass $m$, length $l$, and radius $R$. Use integration as described in Example 8.13 to determine its moment of inertia about the $x$ axis.

Solution: The volume of the disk element is $\pi R^2 dz$ and its mass is $dm = \rho \pi R^2 dz$, where $\rho$ is the density of the cylinder. From Appendix C, the moment of inertia of the disk element about the $x'$ axis is

$$dI_{x' \text{axis}} = \frac{1}{2} dm R^2 = \frac{1}{2} (\rho \pi R^2 dz) R^2.$$

Applying the parallel-axis theorem, the moment of inertia of the disk element about the $x$ axis is

$$dI_{x \text{axis}} = dI_{x' \text{axis}} + z^2 dm = \frac{1}{4} (\rho \pi R^2 dz) R^2 + z^2 (\rho \pi R^2 dz).$$

Integrating this expression from $z = 0$ to $z = l$ gives the moment of inertia of the cylinder about the $x$ axis.

$$I_{x \text{axis}} = \int_0^l \left( \frac{1}{4} \rho \pi R^4 + \rho \pi R^2 z^2 \right) dz = \frac{1}{4} \rho \pi R^4 l + \frac{1}{3} \rho \pi R^4 l^3.$$

In terms of the mass of the cylinder $m = \rho \pi R^4 l$,

$$I_{x \text{axis}} = \frac{1}{2} m R^2 + \frac{1}{3} m l^2.$$

Problem 8.123  The homogeneous cone is of mass $m$. Determine its moment of inertia about the $z$ axis, and compare your result with the value given in Appendix C. (See Example 8.13.)

Solution: The differential mass

$$dm = \left( \frac{m}{V} \right) \pi r^2 dz = \frac{3m}{R^2 h} z^2 dz.$$

The moment of inertia of this disk about the $z$ axis is $\frac{1}{3} m r^2$. The radius varies with $z$.

$$r = \left( \frac{R}{h} \right) z,$$

from which

$$I_{z \text{axis}} = \frac{3m R^2}{2h} \int_0^h z^4 dz = \frac{3m R^2}{2h} \left[ \frac{z^5}{5} \right]_0^h = \frac{3m R^2}{10}.$$
Problem 8.124  Determine the moments of inertia of the homogenous cone in Problem 8.123 about the \( x \) and \( y \) axes, and compare your results with the values given in Appendix C.

Solution: The mass density is \( \rho = \frac{m}{V} = \frac{3m}{\pi R^2 h} \). The differential element of mass is \( dm = \rho r^2 dz \). The moment of inertia of this elemental disk about an axis through its center of mass, parallel to the \( x \) - and \( y \)-axes, is
\[
dI_x = \left( \frac{1}{4} \right) r^2 dm.
\]
Use the parallel axis theorem,
\[
I_x = \int \left( \frac{1}{4} \right) r^2 dm + \int z^2 dm.
\]
Noting that \( r = \frac{R}{h} z \), then
\[
r^2 dm = \rho \left( \frac{\pi R^4}{h^4} \right) z^6 dz.
\]
and
\[
z^2 dm = \rho \left( \frac{\pi R^2}{h^2} \right) z^8 dz.
\]
Substitute:
\[
I_x = \rho \frac{\pi R^4}{4h^6} \int_0^h z^8 dz + \rho \left( \frac{\pi R^2}{h^2} \right) \int_0^h z^6 dz.
\]
Integrating and collecting terms:
\[
I_x = \left( \frac{3mR^2}{4h^5} + \frac{3m}{h^2} \right) \left( \frac{1}{5} \right) h^5 = m \left( \frac{3}{20} R^2 + \frac{3}{5} \right).
\]
By symmetry, \( I_y = I_z \).
Problem 8.125  The mass of the homogeneous wedge is \( m \). Use integration as described in Example 8.13 to determine its moment of inertia about the \( z \) axis. (Your answer should be in terms of \( m, a, b, \) and \( h \).)

Solution: Consider a triangular element of the wedge of thickness \( dz \). The mass of the element is the product of the density \( \rho \) of the material and the volume of the element, \( dm = \rho \frac{1}{2} bh dz \). The moments of inertia of the triangular element about the \( x' \) and \( y' \) axes are given by Eqs. (8.30) and (8.31) in terms of the mass of the element, its triangular area, and the moments of inertia of the triangular area:

\[
dI_{x'\text{axis}} = dm \left( \frac{1}{12} bh^3 \right) = \frac{1}{12} bh^3 dz,
\]

\[
dI_{y'\text{axis}} = dm \left( \frac{1}{4} bh^3 \right) = \frac{1}{4} bh^3 dz.
\]

The moment of inertia of this thin plate about the \( z \) axis is

\[
dI_{z\text{axis}} = dI_{x'\text{axis}} + dI_{y'\text{axis}} = \frac{1}{12} bh^3 dz + \frac{1}{4} bh^3 dz.
\]

Integrating this expression from \( z = 0 \) to \( z = a \) gives the moment of inertia of the wedge about the \( z \) axis:

\[
I_{z\text{axis}} = \int_0^a \left( \frac{1}{12} bh^3 + \frac{1}{4} bh^3 \right) dz = \frac{1}{12} ahb^2 + \frac{1}{4} ahb^2.
\]

In terms of the mass \( m = \frac{1}{2} bh a \),

\[
I_{z\text{axis}} = \frac{1}{6} mbh^2 + \frac{1}{2} mb^2.
\]
Problem 8.126  The mass of the homogeneous wedge is \( m \). Use integration as described in Example 8.13 to determine its moment of inertia about the \( x \) axis. (Your answer should be in terms of \( m, a, b, \) and \( h \).)

Solution:  Consider a triangular element of the wedge of thickness \( dz \). The mass of the element is the product of the density \( \rho \) of the material and the volume of the element, \( dm = \rho \frac{1}{2}abh \). The moments of inertia of the triangular element about the \( x' \) axis is given by Eq. (8.30) in terms of the mass of the element, its triangular area, and the moments of inertia of the triangular area:

\[
dI_{x'\text{axis}} = \frac{dm}{A} I_{x'\text{axis}} = \frac{1}{2} \frac{abh}{h} \left( \frac{1}{3} bh^3 \right) = \frac{1}{36} \rho bh^3 dz.
\]

Applying the parallel-axis theorem, the moment of inertia of the triangular element about the \( x \) axis is:

\[
dI_{x\text{axis}} = dI_{x'\text{axis}} + \left[ z^2 + \left( \frac{1}{3} h \right)^2 \right] dm
\]

\[
= \frac{1}{36} \rho bh^3 dz + [z^2 + \left( \frac{1}{3} h \right)^2] \rho \frac{1}{2}abh dz = \frac{1}{12} \rho bh^3 dz + \frac{1}{2} \rho bhz^2 dz.
\]

Integrating this expression from \( z = 0 \) to \( z = a \) gives the moment of inertia of the wedge about the \( x \) axis:

\[
I_{x\text{axis}} = \int_0^a \left( \frac{1}{12} \rho bh^3 + \frac{1}{2} \rho bhz^2 \right) dz = \frac{1}{12} \rho bh^3 a + \frac{1}{2} \rho bh a^2.
\]

In terms of the mass \( m = \frac{1}{2} \rho bh a \),

\[
I_{x\text{axis}} = \frac{1}{6} m h^2 + \frac{1}{3} ma^2.
\]

Problem 8.127  In Example 8.12, suppose that part of the 3-kg bar is sawed off so that the bar is 0.4 m long and its mass is 2 kg. Determine the moment of inertia of the composite object about the perpendicular axis \( L \) through the center of mass of the modified object.

Solution: The mass of the disk is 2 kg. Measuring from the left end of the rod, we locate the center of mass

\[
\bar{x} = \frac{(2 \text{ kg})(0.2 \text{ m}) + (2 \text{ kg})(0.6 \text{ m})}{(2 \text{ kg}) + (2 \text{ kg})} = 0.4 \text{ m}.
\]

The center of mass is located at the point where the rod and disk are connected. The moment of inertia is

\[
I = \frac{1}{3} (2 \text{ kg})(0.4 \text{ m})^2 + \left( \frac{1}{2} (2 \text{ kg})(0.2 \text{ m})^2 + (2 \text{ kg})(0.2 \text{ m})^2 \right)
\]

\[
I = 0.227 \text{ kg-m}^2.
\]
Problem 8.128  The L-shaped machine part is composed of two homogeneous bars. Bar 1 is tungsten alloy with mass density 14,000 kg/m³, and bar 2 is steel with mass density 7800 kg/m³. Determine its moment of inertia about the x axis.

Solution:  The masses of the bars are

\[ m_1 = (14,000)(0.24)(0.08)(0.04) = 10.75 \text{ kg} \]
\[ m_2 = (7800)(0.24)(0.08)(0.04) = 5.99 \text{ kg} \]

Using Appendix C and the parallel axis theorem the moments of inertia of the parts about the x axis are

\[ I_{x \text{ axis}1} = \frac{1}{12} m_1[(0.04)^2 + (0.24)^2] + m_1(0.12)^2 = 0.2079 \text{ kg-m}^2 \]
\[ I_{x \text{ axis}2} = \frac{1}{12} m_2[(0.04)^2 + (0.08)^2] + m_2(0.04)^2 = 0.0136 \text{ kg-m}^2 \]

Therefore

\[ I_{x \text{ axis}} = I_{x \text{ axis}1} + I_{x \text{ axis}2} = 0.221 \text{ kg-m}^2 \]
**Problem 8.129** The homogeneous object is a cone with a conical hole. The dimensions $R_1 = 2$ cm, $R_2 = 1$ cm, $h_1 = 6$ cm, and $h_2 = 3$ cm. It consists of an aluminum alloy with a density of 2700 kg/m$^3$. Determine its moment of inertia about the $x$ axis.

**Solution:** The density of the material is 

$$\rho = 2700 \text{ kg/m}^3.$$ 

The volume of the conical object without the conical hole is 

$$V_1 = \frac{1}{3} \pi R_1^2 h_1 = \frac{1}{3} \pi (2 \text{ cm})^2 (6 \text{ cm}) = 25.1 \text{ cm}^3.$$ 

The mass of the conical object without the conical hole is $m_1 = \rho V_1 = 0.0678 \text{ kg}$. From Appendix C, the moment of inertia of the conical object without the conical hole about the $x$ axis is 

$$I_{x1} = m_1\left(\frac{3}{5} h_1^2 + \frac{2}{20} R_1^2\right)$$ 

$$= (0.0678 \text{ kg}) \left(\frac{3}{5} (6 \text{ cm})^2 + \frac{3}{20} (2 \text{ cm})^2\right) = 1.505 \text{ kg-cm}^2.$$ 

The volume of the conical hole is 

$$V_2 = \frac{1}{3} \pi R_2^2 h_2 = \frac{1}{3} \pi (1 \text{ cm})^2 (3 \text{ cm}) = 3.14 \text{ cm}^3.$$ 

The mass of the material that would occupy the conical hole is $m_2 = \rho V_2 = 8.478 \times 10^{-3} \text{ kg}$. The $z$ coordinate of the center of mass of the material that would occupy the conical hole is 

$$z = h_2 - h_2 + \frac{3}{4} h_2 = 6 \text{ cm} - 3 \text{ cm} + \frac{3}{4} (3 \text{ cm}) = 5.25 \text{ cm}.$$ 

Using Appendix C and applying the parallel-axis theorem, the moment of inertia of the material that would occupy the conical hole is 

$$I_{x2} = m_2\left(\frac{3}{5} h_2^2 + \frac{3}{20} R_2^2\right) + m_2 z^2 = 0.238 \text{ kg-cm}^2.$$ 

The moment of inertia of the conical object with the conical hole is 

$$I_x = (I_{x1})_1 - I_{x2} = 1.267 \text{ kg-cm}^2.$$ 

$$I_x = 1.267 \text{ kg-cm}^2$$. 

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Problem 8.130  The circular cylinder is made of aluminum (Al) with density 2700 kg/m³ and iron (Fe) with density 7860 kg/m³. Determine its moments of inertia about the x' and y' axes.

Solution: We have ρ_{Al} = 2700 kg/m³, ρ_{Fe} = 7860 kg/m³

We first locate the center of mass

\[ \tau = \frac{\rho_{Al}(\pi(0.1 \text{ m})^2(0.6 \text{ m}) + \rho_{Fe}(\pi(0.1 \text{ m})^2(0.9 \text{ m}))}{\rho_{Al}(\pi(0.1 \text{ m})^2(0.6 \text{ m})) + \rho_{Fe}(\pi(0.1 \text{ m})^2(0.6 \text{ m}))} \]

\[ = 0.747 \text{ m} \]

We also have the masses

\[ m_{Al} = \rho_{Al}(\pi(0.1 \text{ m})^2(0.6 \text{ m})) \]
\[ m_{Fe} = \rho_{Fe}(\pi(0.1 \text{ m})^2(0.6 \text{ m})) \]

Now find the moments of inertia

\[ I_{x'} = \frac{1}{2} m_{Al}(0.1 \text{ m})^2 + \frac{1}{2} m_{Fe}(0.1 \text{ m})^2 = 0.995 \text{ kg m}^2 \]

\[ I_{y'} = m_{Al} \left( \frac{(0.6 \text{ m})^2}{12} + \frac{(0.1 \text{ m})^2}{4} \right) + m_{Al}(\pi - 0.3 \text{ m})^2 \]
\[ + m_{Fe} \left( \frac{(0.6 \text{ m})^2}{12} + \frac{(0.1 \text{ m})^2}{4} \right) + m_{Fe}(0.9 \text{ m} - \pi)^2 \]

\[ = 20.1 \text{ kg m}^2 \]

Problem 8.131  The homogenous half-cylinder is of mass m. Determine its moment of inertia about the axis L through its center of mass.

Solution: The centroid of the half cylinder is located a distance of \( \frac{4R}{3\pi} \) from the edge diameter. The strategy is to use the parallel axis theorem to treat the moment of inertia of a complete cylinder as the sum of the moments of inertia for the two half cylinders. From Problem 8.118, the moment of inertia about the geometric axis for a cylinder is \( I_c = mR^2 \), where m is one half the mass of the cylinder.

By the parallel axis theorem,

\[ I_L = 2 \left( \frac{4R}{3\pi} \right)^2 m + I_{cd} \]

Solve

\[ I_{cd} = \left( \frac{I_L}{2} - \left( \frac{4R}{3\pi} \right)^2 m \right) = \left( \frac{mR^2}{2} - \left( \frac{16}{9\pi} \right) mR^2 \right) \]

\[ = mR^2 \left( \frac{1}{2} - \frac{16}{9\pi} \right) \]

\[ = mR^2 \left( \frac{1}{2} - \frac{16}{9\pi} \right) = 0.31987 mR^2 = 0.32 mR^2 \]
Problem 8.132 The homogeneous machine part is made of aluminum alloy with density $\rho = 2800 \text{ kg/m}^3$. Determine its moment of inertia about the $z$ axis.

**Solution:** We divide the machine part into the 3 parts shown. (The dimension into the page is 0.04 m). The masses of the parts are:

- $m_1 = (2800 \times 0.12 \times 0.08 \times 0.04) = 1.075 \text{ kg}$.
- $m_2 = (2800 \times 0.04^2 \times 0.04) = 0.281 \text{ kg}$.
- $m_3 = (2800 \times 0.02^2 \times 0.04) = 0.141 \text{ kg}$.

Using Appendix C and the parallel axis theorem the moment of inertia of part 1 about the $z$ axis is

$$I_{z, \text{axis}1} = \frac{1}{12} m_1 ((0.08)^2 + (0.12)^2) + m_1 (0.06)^2$$

$$= 0.00573 \text{ kg-m}^2.$$  

The moment of inertia of part 2 about the axis through the center $C'$ that is parallel to the $z$ axis is

$$\frac{1}{2} m_2 r^2 = \frac{1}{2} m_2 (0.04)^2.$$  

The distance along the $x$ axis from $C'$ to the center of mass of part 2 is

$$4(0.04)/3 = 0.0170 \text{ m}.$$  

Therefore, the moment of inertia of part 2 about the $z$ axis through its center of mass that is parallel to the axis is

$$\frac{1}{2} m_2 (0.04)^2 - m_2 (0.0170)^2 = 0.000144 \text{ kg-m}^2.$$  

Using this result, the moment of inertia of part 2 about the $z$ axis is

$$I_{z, \text{axis}2} = 0.000144 + m_2 (0.12 + 0.017)^2 = 0.00543 \text{ kg-m}^2.$$  

The moment of inertia of the material that would occupy the hole 3 about the $z$ axis is

$$I_{z, \text{axis}3} = \frac{1}{2} m_3 (0.02)^2 + m_3 (0.12)^2 = 0.00205 \text{ kg-m}^2.$$  

Therefore $I_{z, \text{axis}} = I_{z, \text{axis}1} + I_{z, \text{axis}2} - I_{z, \text{axis}3}$

$$= 0.00911 \text{ kg-m}^2.$$  

Problem 8.133 Determine the moment of inertia of the machine part in Problem 8.132 about the $x$ axis.

**Solution:** We divide the machine part into the 3 parts shown in the solution to Problem 8.132. Using Appendix C and the parallel axis theorem, the moments of inertia of the parts about the $x$ axis are:

- $I_{x, \text{axis}1} = \frac{1}{12} m_1 ((0.08)^2 + (0.04)^2) = 0.0000166 \text{ kg-m}^2$.
- $I_{x, \text{axis}2} = m_2 \left[ \frac{1}{12} (0.04)^2 + \frac{1}{4} (0.04)^2 \right]$
  $$= 0.0000501 \text{ kg-m}^2.$$  

- $I_{x, \text{axis}3} = m_3 \left[ \frac{1}{12} (0.04)^2 + \frac{1}{4} (0.02)^2 \right]$
  $$= 0.0000328 \text{ kg-m}^2.$$  

Therefore, $I_{x, \text{axis}} = I_{x, \text{axis}1} + I_{x, \text{axis}2} - I_{x, \text{axis}3}$

$$= 0.0000834 \text{ kg-m}^2.$$  

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**Problem 8.134**  The object consists of steel of density $\rho = 7800 \text{ kg/m}^3$. Determine its moment of inertia about the axis $L_O$.

**Solution:** Divide the object into four parts: Part (1) The semi-cylinder of radius $R = 0.02 \, \text{m}$, height $h_1 = 0.01 \, \text{m}$.

Part (2): The rectangular solid $L = 0.1 \, \text{m}$ by $h_2 = 0.01 \, \text{m}$ by $w = 0.04 \, \text{m}$. Part (3): The semi-cylinder of radius $R = 0.02 \, \text{m}$, $h_1 = 0.01 \, \text{m}$. Part (4) The cylinder of radius $R = 0.02 \, \text{m}$, height $h = 0.03 \, \text{m}$.

Part (1) 

$m_1 = \frac{\rho \pi R^2 h_1}{2} = 0.049 \, \text{kg}$,

$I_1 = \frac{m_1 R^2}{4} = 4.9 \times 10^{-6} \, \text{kg-m}^2$.

Part (2):

$m_2 = \rho w L h_2 = 0.312 \, \text{kg}$,

$I_2 = \left( \frac{1}{12} \right) m_2 (L^2 + w^2) + m_2 \left( \frac{L}{2} \right)^2 = 0.00108 \, \text{kg-m}^2$.

Part (3)

$m_3 = m_1 = 0.049 \, \text{kg}$,

$I_3 = -\left( \frac{4R}{3\pi} \right)^2 m_2 + I_1 + m_3 \left( L - \frac{4R}{3\pi} \right)^2 = 0.00041179 \, \text{kg-m}^2$.

Part (4)

$m_4 = \rho \pi R^2 h = 0.294 \, \text{kg}$,

$I_4 = \left( \frac{1}{2} \right) m_4 R^2 + m_4 L^2 = 0.003 \, \text{kg-m}^2$.

The composite:

$I_{LO} = I_1 + I_2 - I_3 + I_4 = 0.003674 \, \text{kg-m}^2$

**Problem 8.135** Determine the moment of inertia of the object in Problem 8.134 about the axis through the center of mass of the object parallel to $L_O$.

**Solution:** The center of mass is located relative to $L_O$

$x = \frac{m_1 \left( \frac{4R}{3\pi} \right) + m_2 (0.05) - m_3 \left( 0.1 - \frac{4R}{3\pi} \right) + m_4 (0.1)}{m_1 + m_2 - m_3 + m_4} = 0.066 \, \text{m},$

$I_c = -x^2 m + I_{LO} = -0.00265 + 0.00367 = 0.00102 \, \text{kg-m}^2$
**Problem 8.136**  The thick plate consists of steel of density \( \rho = 7800 \text{ kg/m}^3 \). Determine its moment of inertia about the \( z \) axis.

**Solution:**  Divide the object into three parts: Part (1) the rectangle 8 cm by 16 cm, Parts (2) & (3) the cylindrical cut outs. Part (1):

\[ m_1 = \rho (0.16 \times 0.04) = 3.994 \text{ kg}. \]

\[ I_1 = \left( \frac{1}{12} \right) m_1 (0.16^2 + 0.08^2) = 0.01065 \text{ kg-m}^2. \]

Part (2):

\[ m_2 = \rho \pi (0.02^2)(0.04) = 0.3921 \text{ kg}. \]

\[ I_2 = \frac{m_2 (0.02^2)}{2} + m_2 (0.04^2) = 0.0007057 \text{ kg-m}^2. \]

Part (3):

\[ m_3 = m_2 = 0.3921 \text{ kg}. \]

\[ I_3 = I_2 = 0.0007057 \text{ kg-m}^2. \]

The composite:

\[ I_z = I_1 - 2I_2 = 0.00924 \text{ kg-m}^2. \]

---

**Problem 8.137**  Determine the moment of inertia of the plate in Problem 8.136 about the \( x \) axis.

**Solution:**  Use the same divisions of the object as in Problem 8.136.

Part (1):

\[ I_{x \text{ axis}} = \left( \frac{1}{12} \right) m_1 (0.08^2 + 0.04^2) = 0.002662 \text{ kg-m}^2. \]

Part (2):

\[ I_{x \text{ axis}} = \left( \frac{1}{12} \right) m_2 (3(0.02^2) + 0.04^2) = 9.148 \times 10^{-5} \text{ kg-m}^2. \]

The composite:

\[ I_x = I_{x \text{ axis}} - 2I_{x \text{ axis}} = 0.00248 \text{ kg-m}^2. \]
Problem 8.138  Determine $I_y$ and $k_y$.

Solution:

\[ dA = dx \, dy \Rightarrow A = \int_0^1 dx \int_0^{x^2} dy = \int_0^1 x^2 \, dx = \frac{1}{3} \]

\[ I_y = \int_A x^2 \, dA = \int_0^1 x^2 \, dx \int_0^{x^2} dy = \int_0^1 x^4 \, dx = \left[ \frac{x^5}{5} \right]_0^1 = \frac{1}{5} \]

\[ k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{1}{5}} \]

Problem 8.139  Determine $I_x$ and $k_x$.

Solution:  (See figure in Problem 8.138.) $dA = dx \, dy$,

\[ I_x = \int_A y^2 \, dA = \int_0^1 dy \int_0^{x^2} y^2 \, dx = \frac{1}{3} \int_0^1 x^5 \, dx \]

\[ = \left( \frac{1}{21} \right) [x^7]_0^1 = \frac{1}{21} \]

\[ k_x = \sqrt{\frac{I_x}{A}} = \frac{1}{\sqrt{7}} \]

Problem 8.140  Determine $J_O$ and $k_O$.

Solution:  (See figure in Problem 8.138.)

\[ J_O = I_x + I_y = \frac{1}{5} + \frac{1}{21} = \frac{26}{105} \]

\[ k_O = \sqrt{k_x^2 + k_y^2} = \sqrt{\left( \frac{1}{\sqrt{7}} \right)^2 + \left( \frac{1}{\sqrt{5}} \right)^2} = \frac{2\sqrt{35}}{35} \]

Problem 8.141  Determine $I_{xy}$.

Solution:  (See figure in Problem 8.138.) $dA = dx \, dy$

\[ I_{xy} = \int_A xy \, dA = \int_0^1 x \, dx \int_0^{x^2} y \, dy \]

\[ = \frac{1}{2} \int_0^1 x^5 \, dx = \frac{1}{12} \left[ x^6 \right]_0^1 = \frac{1}{12} \]
Problem 8.142  Determine $I_y$ and $k_y$.

Solution:  By definition,

$$I_y = \int_A x^2 \, dA.$$  

The element of area is $dA = dx \, dy$. The limits on the variable $x$ are $0 \leq x \leq 4$. The area is

$$A = \int_0^4 dx \int_0^{x-x^2/4} dy = \left[ \frac{x^2}{2} - \frac{x^3}{12} \right]_0^4 = 2.6667$$

$$I_y = \int_0^4 x^2 \, dx \int_0^{x-x^2/4} dy = \int_0^4 \left( x - \frac{x^2}{4} \right)^2 \, dx$$

$$= \left[ \frac{x^4}{4} - \frac{x^5}{20} \right]_0^4 = 12.8$$

from which

$$k_y = \frac{I_y}{A} = 2.19$$

Problem 8.143  Determine $I_x$ and $k_x$.

Solution:  By definition,

$$I_x = \int_A y^2 \, dA.$$  

from which

$$I_x = \int_0^4 dx \int_0^{x-x^2/4} y^2 \, dy = \left( \frac{1}{3} \right) \int_0^4 \left( x - \frac{x^2}{4} \right)^3 \, dx$$

$$I_x = \left( \frac{1}{3} \right) \left[ \frac{x^4}{4} - \frac{3}{20} x^5 + \frac{3}{160} x^6 - \frac{x^7}{448} \right]_0^4 = 0.6095.$$  

From Problem 8.142,

$$A = 2.667, \quad k_x = \sqrt{\frac{I_x}{A}} = 0.4781$$

Problem 8.144  Determine $I_{xy}$.

Solution:

$$I_{xy} = \int_A xy \, dA,$$

$$= \int_0^4 dx \int_0^{x-x^2/4} xy \, dy$$

$$= \left( \frac{1}{2} \right) \int_0^4 \left( x - \frac{x^2}{4} \right)^2 \, dx$$

$$= \left( \frac{1}{2} \right) \left[ \frac{x^4}{4} - \frac{x^5}{10} + \frac{x^6}{36} \right]_0^4 = 2.1333$$

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Problem 8.145 Determine $I_{y'}$ and $k_{y'}$.

Solution: The limits on the variable $x$ are $0 \leq x \leq 4$. By definition,

$$Ay = \int_A y \, dA = \int_0^4 dx \int_{-x^2/4}^{x^2/4} y \, dy$$

$$= \left( \frac{1}{2} \right) \int_0^4 \left( x - \frac{x^2}{4} \right)^2 \, dx$$

$$= \left( \frac{1}{2} \right) \left[ \frac{x^3}{3} - \frac{x^4}{8} \right]_0^4 = 1.06667.$$

From Problem 8.142 the area is $A = 2.667$, from which $y = 0.3999 = 0.4$. Similarly,

$$Ax = \int_0^4 x \, dx \int_{-x^2/4}^{x^2/4} y \, dy$$

$$= \int_0^4 x \left( x - \frac{x^2}{4} \right) \, dx = \left[ \frac{x^3}{3} - \frac{x^4}{16} \right]_0^4 = 5.3333,$$

from which $x = 1.9999 = 2$. The area moment of inertia is $I_{y'} = -x^2 A + I_x$. Using the result of Problem 8.142, $I_x = 12.8$, from which the area moment of inertia about the centroid is

$$I_{y'} = -10.6666 + 12.8 = 2.133$$

and $k_{y'} = \sqrt{\frac{I_{y'}}{A}} = 0.8944$

Problem 8.146 Determine $I_{x'}$ and $k_{x'}$.

Solution: Using the results of Problems 8.143 and 8.145, $I_x = 0.6095$ and $y = 0.4$. The moment of inertia about the centroid is

$$I_{x'} = -y^2 A + I_x = 0.1828$$

and $k_{x'} = \sqrt{\frac{I_{x'}}{A}} = 0.2618$

Problem 8.147 Determine $I_{x'y'}$.

Solution: From Problems 8.143 and 8.144, $I_{xy} = 2.133$ and $x = 2$, $y = 0.4$. The product of the moment of inertia about the centroid is

$$I_{x'y'} = -xy A + I_{xy} = -2.133 + 2.133 = 0$$
Problem 8.148  Determine $I_y$ and $k_y$.

Solution: Divide the section into two parts: Part (1) is the upper rectangle 40 mm by 200 mm, Part (2) is the lower rectangle, 160 mm by 40 mm.

Part (1): $A_1 = 0.040(0.200) = 0.008 \text{ m}^2$,

$$y_1 = 0.180 \text{ m},$$

$$x_1 = 0,$$

$$I_{1y} = \left(\frac{1}{12}\right)0.04(0.2)^2 = 2.6667 \times 10^{-5} \text{ m}^4.$$

Part (2): $A_2 = (0.04)(0.16) = 0.0064 \text{ m}^2$,

$$y_2 = 0.08 \text{ m},$$

$$x_2 = 0,$$

$$I_{2y} = \left(\frac{1}{12}\right)(0.16)(0.04)^3 = 8.5 \times 10^{-7} \text{ m}^4.$$

The composite:

$$A = A_1 + A_2 = 0.0144 \text{ m}^2,$$

$$I_y = I_{1y} + I_{2y},$$

$$I_y = 2.752 \times 10^{-5} \text{ m}^4 = 2.752 \times 10^7 \text{ mm}^4,$$

and $k_y = \sqrt{\frac{I_y}{A}} = 0.0437 \text{ m} = 43.7 \text{ mm}$

Problem 8.149  Determine $I_x$ and $k_x$ for the area in Problem 8.148.

Solution: Use the results in the solution to Problem 8.148. Part (1)

$$A_1 = 0.040(0.200) = 0.008 \text{ m}^2,$$

$$y_1 = 0.180 \text{ m},$$

$$I_{1x} = \left(\frac{1}{12}\right)0.2(0.04^3) + (0.18)^2A_1 = 2.603 \times 10^{-4} \text{ m}^4.$$

Part (2):

$$A_2 = (0.04)(0.16) = 0.0064 \text{ m}^2,$$

$$y_2 = 0.08 \text{ m},$$

$$I_{2x} = \left(\frac{1}{12}\right)(0.04)(0.16)^3 + (0.08)^2A_2 = 5.461 \times 10^{-5} \text{ m}^4.$$

The composite: $A = A_1 + A_2 = 0.0144 \text{ m}^2$. The area moment of inertia about the $x$ axis is

$$I_x = I_{1x} + I_{2x} = 3.15 \times 10^{-4} \text{ m}^4 = 3.15 \times 10^8 \text{ mm}^4,$$

and $k_x = \sqrt{\frac{I_x}{A}} = 0.1479 \text{ m} = 147.9 \text{ mm}$

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Problem 8.150  Determine $I_x$ and $k_x$.

Solution: Use the results of the solutions to Problems 8.148–8.149. The centroid is located relative to the base at

$$x = \frac{x_1 A_1 + x_2 A_2}{A} = 0,$$

$$y = \frac{y_1 A_1 + y_2 A_2}{A} = 0.1356 \text{ m}.$$

The moment of inertia about the $x$ axis is

$$I_{xw} = -y^2_A A + I_x = 5.028 \times 10^7 \text{ mm}^4$$

and $k_{xw} = \sqrt{\frac{I_{xw}}{A}} = 59.1 \text{ mm}$

Problem 8.151  Determine $J_O$ and $k_O$ for the area in Problem 8.150.

Solution: Use the results of the solutions to Problems 8.148–8.149. The area moments of inertia about the centroid are

$$I_{xw} = 5.028 \times 10^{-5} \text{ m}^4$$

and $I_{yw} = I_y = 2.752 \times 10^{-5} \text{ m}^4$.

from which

$$J_O = I_{xw} + I_{yw} = 7.78 \times 10^{-5} \text{ m}^4 = 7.78 \times 10^7 \text{ mm}^4$$

and $k_O = \sqrt{\frac{J_O}{A}} = 0.0735 \text{ m}$

$$= 73.5 \text{ mm}$$

Problem 8.152  Determine $I_y$ and $k_y$.

Solution: For a semicircle about a diameter:

$$I_{yy} = I_{xx} = \left(\frac{1}{8}\right) \pi R^4,$$

$$I_y = \left(\frac{1}{8}\right) \pi (4^4) - \left(\frac{1}{8}\right) \pi (2^4) = \frac{\pi}{8}(4^4 - 2^4) = 94.25 \text{ cm}^4.$$ 

$$k_y = \sqrt{\frac{2I_y}{\pi(4^4 - 2^4)}} = 2.236 \text{ cm}$$
**Problem 8.153** Determine \( J_O \) and \( k_O \) for the area in Problem 8.152.

**Solution:** For a semicircle:

\[
I_{yy} = I_{xx} = \left( \frac{1}{8} \right) \pi R^4.
\]

\[
I_s = \frac{\pi}{8} (R^4 - 2a^4) = 94.248 \text{ cm}^4.
\]

\[
k_s = \sqrt{\frac{2I_s}{\pi (R^4 - 2a^4)}} = 2.236 \text{ cm}.
\]

Also use the solution to Problem 8.152.

\[
J_O = I_s + I_y = 2(94.248) = 188.5 \text{ cm}^4
\]

\[
k_O = \sqrt{\frac{2J_O}{\pi (R^4 - 2a^4)}} = 3.16 \text{ cm}
\]

**Problem 8.154** Determine \( I_x \) and \( k_x \).

**Solution:** Break the area into three parts: Part (1) The rectangle with base \( 2a \) and altitude \( h \); Part (2) The triangle on the right with base \( b - a \) and altitude \( h \), and Part (3) The triangle on the left with base \( b - a \) and altitude \( h \). Part (1) The area is

\[
A_1 = 2ah = 24 \text{ m}^2.
\]

The centroid is

\[
x_1 = 0
\]

and \( y_1 = \frac{h}{2} = 3 \text{ m} \).

The area moment of inertia about the centroid is

\[
I_{x1} = \left( \frac{1}{12} \right) 2a h^3 = \left( \frac{1}{6} \right) ah^3 = 72 \text{ m}^4.
\]

Part (2): \( A_2 = \left( \frac{1}{2} \right) h (b - a) = 3 \text{ m}^2 \).

\[
x_2 = a + \frac{b - a}{3} = 2.3333 \text{ m}.
\]

\[
y_2 = \frac{2}{3} h = 4 \text{ m}.
\]

\[
I_{x2} = \left( \frac{1}{36} \right) (b - a) h^3 = 6 \text{ m}^4.
\]

Part (3): \( A_3 = A_1 \).

\[
x_3 = -x_2, \quad y_3 = y_2, \quad I_{x3} = I_{x2}.
\]

The composite area is

\[
A = A_1 + A_2 + A_2 = 30 \text{ m}^2.
\]

The composite moment of inertia

\[
I_x = (y_1)^2 A_1 + (y_2)^2 A_2 + (y_3)^2 A_3 + I_{x1} + I_{x2} + I_{x3}.
\]

\[
I_x = 396 \text{ m}^4
\]

\[
k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{396}{30}} = 3.633 \text{ m}.
\]
Problem 8.155  Determine $I_y$ and $k_y$ for the area in Problem 8.154.

Solution:  Divide the area as in the solution to Problem 8.154.
Part (1) The area is $A_1 = 2ab = 24 \text{ m}^2$. The centroid is $x_1 = 0$ and $y_1 = \frac{b}{2} = 3 \text{ m}$. The area moment of inertia about the centroid is

$$I_{y1} = \left(\frac{1}{12}\right) ah^3 = \left(\frac{2}{3}\right) ah^3 = 32 \text{ m}^4$$

Part (2): $A_2 = \left(\frac{1}{2}\right) h(b - a) = 3 \text{ m}^2$,

$$x_2 = a + \frac{b - a}{3} = 2.3333 \text{ m},$$

$$y_2 = \left(\frac{2}{3}\right) h = 4 \text{ m},$$

$$I_{y2} = \left(\frac{1}{36}\right) h(b - a)^3 = 0.1667 \text{ m}^4.$$  

Part (3): $A_3 = A_2$,

$$x_3 = -x_2, \quad y_3 = y_2; \quad I_{y3} = I_{y2}.$$  
The composite area is $A = A_1 + A_2 + A_3 = 30 \text{ m}^2$.

The composite moment of inertia,

$$I_y = x_2^2 A_2 + I_{y1} + x_3^2 A_3 + I_{y2} + x_3^2 A_3 + I_{y3},$$

$$I_y = 65 \text{ m}^4$$

and $k_y = \sqrt{\frac{I_y}{A}} = 1.472 \text{ m}$

Problem 8.156  The moments of inertia of the area are $I_x = 36 \text{ m}^4$, $I_y = 145 \text{ m}^4$, and $I_{xy} = 44.25 \text{ m}^4$. Determine a set of principal axes and the principal moment of inertia.

Solution:  The principal angle is

$$\theta = \left(\frac{1}{2}\right) \tan^{-1}\left(\frac{2I_{xy}}{I_y - I_x}\right) = 19.54^\circ.$$  
The principal moments of inertia are

$$I_{xP} = I_x \cos^2 \theta - 2I_{xy} \sin \theta \cos \theta + I_y \sin^2 \theta = 20.298 = 20.3 \text{ m}^4$$

$$I_{yP} = I_x \sin^2 \theta + 2I_{xy} \sin \theta \cos \theta + I_y \cos^2 \theta = 160.70 \text{ m}^4$$
Problem 8.157  The moment of inertia of the 0.88 kg bat about a perpendicular axis through point B is 0.122 kg-m^2. What is the bat’s moment of inertia about a perpendicular axis through point A? (Point A is the bat’s “instantaneous center,” or center of rotation, at the instant shown.)

Solution:  Use the parallel axis theorem to obtain the moment of inertia about the center of mass C, and then use the parallel axis theorem to translate to the point A.

\[ I_C = -(0.3)^2 m = 0.04280 \text{ kg-m}^2 \]

\[ I_A = (0.3 + 0.35)^2 m + 0.04280 = 0.415 \text{ kg-m}^2 \]

---

Problem 8.158  The mass of the thin homogenous plate is 4 kg. Determine its moment of inertia about the y axis.

Solution:  Divide the object into two parts: Part (1) is the semi-circle of radius 100 mm, and Part (2) is the rectangle 200 mm by 280 mm. The area of Part (1)

\[ A_1 = \frac{\pi R^2}{2} = 15708 \text{ mm}^2. \]

The area of Part (2) is

\[ A_2 = 280(200) = 56000 \text{ mm}^2. \]

The composite area is \( A = A_2 - A_1 = 40292 \text{ mm}^2 \). The area mass density is

\[ \rho = \frac{m}{A} = 9.9275 \times 10^{-5} \text{ kg/mm}^2. \]

For Part (1) \( x_1 = y_1 = 0 \),

\[ I_{11} = \rho \left( \frac{1}{8} \right) \pi R^4 = 3898.5 \text{ kg-mm}^2. \]

For Part (2) \( x_2 = 100 \text{ mm} \),

\[ I_{22} = x_2^2 \rho A_2 + \rho \left( \frac{1}{12} \right) (280)(200^2) = 74125.5 \text{ kg-mm}^2. \]

The composite:

\[ I_y = I_{22} - I_{11} = 70226 \text{ kg-mm}^2 = 0.070226 \text{ kg-m}^2 \]
Problem 8.159  
Determine the moment of inertia of the plate in Problem 8.158 about the $z$ axis.

Solution:  
Use the same division of the parts and the results of the solution to Problem 8.158. For Part (1),

$$I_{x1} = \rho \left( \frac{1}{8} \right) \pi R^4 = 3898.5 \text{ kg-mm}^2.$$  

For Part (2)

$$I_{x2} = \rho \left( \frac{1}{12} \right) (200 \text{ mm}) (280 \text{ mm})^3 = 36321.5 \text{ kg-mm}^2.$$  

The composite: $I_x = I_{x2} - I_{x1} = 3242 \text{ kg-mm}^2$.

Problem 8.160  
The homogenous pyramid is of mass $m$. Determine its moment of inertia about the $z$ axis.

Solution:  
The mass density is

$$\rho = \frac{m}{V} = \frac{3m}{w^2 h}.$$  

The differential mass is $dm = \rho \omega^2 dz$. The moment of inertia of this element about the $z$ axis is

$$dI_z = \left( \frac{1}{12} \right) \omega^2 dm.$$  

Noting that $\omega = \frac{w}{h}$, then

$$dI_z = \rho \left( \frac{w^4}{24h^3} \right) z^4 dz = \frac{m w^2}{2h} z^4 dz.$$  

Integrating:

$$I_{z, \text{ axis}} = \left( \frac{m w^2}{2h^4} \right) \int_0^h z^4 dz = \frac{1}{10} m w^2.$$  

Problem 8.161  
Determine the moment of inertia of the homogenous pyramid in Problem 8.160 about the $x$ and $y$ axes.

Solution:  
Use the results of the solution of Problem 8.160 for the mass density. The elemental disk is $dm = \rho \omega^2 dz$. The moment of inertia about an axis through its center of mass parallel to the $x$ axis is

$$dI_x = \left( \frac{1}{12} \right) \omega^2 dm.$$  

Use the parallel axis theorem:

$$I_{x, \text{ axis}} = \left( \frac{1}{12} \right) \int_m \omega^2 dm + \int_m z^2 \omega^2 dz.$$  

Noting that $\omega = \frac{w}{h}$, the integral is

$$I_{x, \text{ axis}} = \frac{m w^2}{12h^3} \int_0^h z^4 dz + \frac{m w^2}{h^3} \int_0^h z^2 dz.$$  

Integrating and collecting terms

$$I_{x, \text{ axis}} = m \left( \frac{1}{20} w^4 + \frac{3}{5} z^2 \right).$$  

By symmetry, $I_{y, \text{ axis}} = I_{x, \text{ axis}}$. 

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Problem 8.162  The homogenous object weighs 400 N. Determine its moment of inertia about the \( x \) axis.

Solution:  The volumes are

\[
V_{cyl} = (46)\pi(9)^2 = 11,706 \text{ cm}^3, \\
V_{cone} = \frac{1}{3}(6)^2(36) = 1357 \text{ cm}^3,
\]

so \( V = V_{cyl} - V_{cone} = 10,348 \text{ cm}^3 \).

The masses of the solid cylinder and the material that would occupy the conical hole are

\[
m_{cyl} = \left( \frac{V_{cyl}}{V} \right) \left( \frac{400}{9.81} \right) = 46.122 \text{ kg}, \\
m_{cone} = \left( \frac{V_{cone}}{V} \right) \left( \frac{400}{9.81} \right) = 5.348 \text{ kg}.
\]

Using results from Appendix C,

\[
I_{(x \text{ axis})} = \frac{1}{2} m_{cyl}(9)^2 - \frac{3}{10} m_{cone}(6)^2 \\
= 1810 \text{ kg-cm}^2
\]

Problem 8.163  Determine the moments of inertia of the object in Problem 8.162 about the \( y \) and \( z \) axes.

Solution:  See the solution of Problem 8.162. The position of the center of mass of the material that would occupy the conical hole is

\[
x = (46 - 36) + \frac{3}{4}(36) = 37 \text{ cm}.
\]

From Appendix C,

\[
I_{(y \text{ axis}) \text{, \ cone}} = m_{cone} \left[ \frac{3}{80}(36)^2 + \frac{3}{20}(6)^2 \right] \\
= 288.77 \text{ kg-cm}^2.
\]

The moment of inertia about the \( y \) axis for the composite object is

\[
I_{(y \text{ axis}) \text{, \ object}} = m_{cyl} \left[ \frac{1}{4}(46)^2 + \frac{1}{4}(9)^2 \right] \\
- \left( I_{(y \text{ axis}) \text{, \ cone}} + x m_{cone} \right) \\
= 25856 \text{ kg-cm}^2
\]
Problem 8.164  Determine the moment of inertia of the 14-kg flywheel about the axis \( L \).

Solution:  The flywheel can be treated as a composite of the objects shown:

The volumes are
\[
V_1 = (150)\pi(250)^2 = 294.5 \times 10^5 \text{ mm}^3, \\
V_2 = (150)\pi(220)^2 = 228.08 \times 10^5 \text{ mm}^3, \\
V_3 = (50)\pi(220)^2 = 76.03 \times 10^5 \text{ mm}^3, \\
V_4 = (50)\pi(60)^2 = 5.65 \times 10^5 \text{ mm}^3, \\
V_5 = (100)\pi(60)^2 = 11.31 \times 10^5 \text{ mm}^3, \\
V_6 = (100)\pi(35)^2 = 3.85 \times 10^5 \text{ mm}^3.
\]

The volume
\[
V = V_1 - V_2 + V_3 - V_4 + V_5 - V_6 \\
= 144.3 \times 10^5 \text{ mm}^3,
\]
so the density is
\[
\delta = \frac{14}{V} = 9.704 \times 10^{-7} \text{ kg/mm}^3.
\]

The moment of inertia is
\[
I_L = \frac{1}{4}V_1(250)^2 - \frac{1}{4}V_2(220)^2 \\
+ \frac{1}{4}V_3(220)^2 - \frac{1}{2}V_4(60)^2 \\
+ \frac{1}{4}V_5(60)^2 - \frac{1}{4}V_6(35)^2 \\
= 536,800 \text{ kg-mm}^2 \\
= 0.5368 \text{ kg-m}^2.
\]