Solutions: Exercises Functions

Exercise 0.1.7 In each case determine whether $f$ is well-defined. Give reason for your answer.

1. $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(n) = -n$ for all $n \in \mathbb{Z}$.
2. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$ for all $x \in \mathbb{R}$.
3. $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x, y) = 2x + 3y$ for all $x, y \in \mathbb{R}$.
4. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}$.
5. $f : \mathbb{Q} \rightarrow \mathbb{Z}$ defined by $f(\frac{m}{n}) = mn$ for all $m, n \in \mathbb{Z}$, $n \neq 0$.

Solutions:

1. If $n, m \in \mathbb{Z}$ such that $n = m$, then $f(n) = -n = -m = f(m)$. So $f$ is well-defined.

2. Since $-1 \in \mathbb{R}$ and $\sqrt{-1} \notin \mathbb{R}$, $f$ is not defined. So $f$ is not a function.

3. Let $(x, y)$ and $(\hat{x}, \hat{y})$ be two elements of $\mathbb{R} \times \mathbb{R}$ such that $(x, y) = (\hat{x}, \hat{y})$. Then $x = \hat{x}$ and $y = \hat{y}$. Now $f(x, y) = 2x + 3y = 2\hat{x} + 3\hat{y} = f(\hat{x}, \hat{y})$. Hence $f$ is well-defined.

4. Since $0 \in \mathbb{R}$ and $\frac{1}{0} \notin \mathbb{R}$, $f$ is not defined. So $f$ is not a function.

5. $f$ is not well-defined, because for example $\frac{2}{3} = \frac{4}{6}$ but $f(\frac{2}{3}) = 6 \neq f(\frac{4}{6}) = 24$.

Exercise 0.1.8 In each case determine whether the indicated function is onto, one-to-one, or bijective. Justify your answer.

1. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2 - 3x$ for all $x \in \mathbb{R}$.
2. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x^2 + 3$ for all $x \in \mathbb{R}$.
3. $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 3}$ defined by $f(x) = 2x^2 + 3$ for all $x \in \mathbb{R}$.
4. $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(n) = n^2 + n$ for all $n \in \mathbb{Z}$.
5. $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) = \left\{ \begin{array}{ll} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{array} \right.$

Solutions:
1. If \( f(x) = f(y) \), then \( 2 - 3x = 2 - 3y \). So \( 3x = 3y \) and \( x = y \). Hence \( f \) is one-to-one. (1)

If \( y \in \mathbb{R} \), then we can find \( x \in \mathbb{R} \) such that \( f(x) = y \): For this we need to have \( 2 - 3x = y \) which implies \( x = \frac{2 - y}{3} \). Now we can see that

\[
f(x) = f\left(\frac{2 - y}{3}\right) = 2 - 3\left(\frac{2 - y}{3}\right) = 2 - 2 + y = y.
\]

Hence \( f \) is onto. (2). Therefore by (1) and (2) \( f \) is a bijection.

2. \( f \) is not one-to-one: Because for example \( f(1) = f(-1) = 5 \), but \( 1 \neq -1 \).

It is not difficult to see that

\[
\text{Im}(f) = \{ f(x) \mid x \in \mathbb{R} \} = \{ 2x^2 + 3 \mid x \in \mathbb{R} \} = \mathbb{R}_{\geq 3} \neq \mathbb{R}.
\]

So \( f \) is not onto.

3. By part 2 above we have \( f \) is not one-to-one. But \( \text{Im}(f) = \mathbb{R}_{\geq 3} \) implies that \( f \) is onto.

4. Since \( f(0) = f(-1) = 0 \) and \( 0 \neq -1 \), \( f \) is not one-to-one. \( f \) is not onto since for example \( 1 \in \mathbb{Z} \) but we can not find \( n \in \mathbb{Z} \) such that \( f(n) = n^2 + n = 1 \). Note that the equation \( n^2 + n - 1 = 0 \) has no integer solution.

5. Since

\[
f(1) = \frac{1+1}{2} = 1 \quad \text{and} \quad f(2) = \frac{2}{2} = 1
\]

we have \( f(1) = f(2) \). But \( 1 \neq 2 \) implies that \( f \) is not one-to-one.

Let \( m \) be a natural number and let \( n = 2m \). Then \( f(n) = \frac{2m}{2} = m \). So \( f \) is onto.

**Exercise 0.1.9** Let \( X \) and \( Y \) be two sets of three and two elements respectively. Show that there are eight functions from \( X \) into \( Y \) and nine functions from \( Y \) into \( X \). How many functions from \( X \) onto \( Y \) are there?

**Solutions:** Let \( X = \{a, b, c\} \) and \( Y = \{x, y\} \). Then the functions from \( X \) into \( Y \) are:

\[
\begin{align*}
&f_1 : a \to x, b \to x, c \to x & f_2 : a \to y, b \to y, c \to y \\
&f_3 : a \to x, b \to x, c \to y & f_4 : a \to y, b \to y, c \to x \\
&f_5 : a \to y, b \to x, c \to x & f_6 : a \to y, b \to x, c \to y \\
&f_7 : a \to x, b \to y, c \to y & f_8 : a \to y, b \to y, c \to x \\
\end{align*}
\]
It is easy to see that \( f_3, f_4, f_5, f_6, f_7 \) and \( f_8 \) are onto.

The functions from \( Y \) into \( X \) are:

\[
\begin{align*}
g_1 : & \quad x \to a, y \to a \\
g_2 : & \quad x \to b, y \to b \\
g_3 : & \quad x \to c, y \to c \\
g_4 : & \quad x \to a, y \to b \\
g_5 : & \quad x \to a, y \to c \\
g_6 : & \quad x \to b, y \to a \\
g_7 : & \quad x \to b, y \to c \\
g_8 : & \quad x \to c, y \to a \\
g_9 : & \quad x \to c, y \to b \\
\end{align*}
\]

**Exercise 0.1.10** Let \( \mathbb{Z}^* \) denote the set of nonzero integers. Consider \( f : \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{Q} \) defined by \( f(n, m) = \frac{n}{m} \). Show that \( f \) is a function. Determine whether \( f \) is one-to-one or onto.

**Solution:** If \((n, m) = (\hat{n}, \hat{m})\), then \( n = \hat{n} \) and \( m = \hat{m} \). Hence \( f(n, m) = \frac{n}{m} = \frac{\hat{n}}{\hat{m}} = f(\hat{n}, \hat{m}) \). So \( f \) is well-defined.

\( f \) is not one-to-one: Because for example \( f(2, 3) = f(4, 6) \), but \( 2, 3 \neq (4, 6) \).

\( f \) is onto, since 

\[
\text{Im}(f) = \left\{ \frac{n}{m} \mid n, m \in \mathbb{Z}, m \neq 0 \right\} = \mathbb{Q}.
\]

**Exercise 0.1.11** Consider the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = b + ax \) for all \( x \in \mathbb{R} \) where \( a \neq 0 \) and \( b \) are constant real numbers. Show that \( f \) is invertible and find \( f^{-1} \).

**Solution:** Similar to Ex.0.1.7 we can show that \( f \) is a bijection. Hence \( f \) is invertible. If \( f^{-1}(x) = y \), then \( f(y) = x \). That is \( b + ay = x \), and hence 

\[
y = \frac{1}{a}x - \frac{b}{a}.
\]

So \( f^{-1}(x) = \frac{1}{a}x - \frac{b}{a} \). It is easy to check that \( f \circ f^{-1} = f^{-1} \circ f = 1_{\mathbb{R}} \).

**Exercise 0.1.12** Show that if \( f : X \to Y \) and \( g : Y \to Z \) are functions, then \( g \circ f : X \to Z \) is also a function.

**Solution:** \( g \circ f : X \to Y \). Let \( x_1, x_2 \in X \) such that \( x_1 = x_2 \). Then since \( f \) is a function, we have \( f(x_1) = f(x_2) \). Now since \( g \) is function and \( f(x_1) = f(x_2) \), we must have \( g(f(x_1)) = g(f(x_2)) \). That is \( (g \circ f)(x_1) = (g \circ f)(x_2) \). Hence \( g \circ f \) is well-defined.

**Exercise 0.1.13** If \( f : X \to Y \) is a function, show that \( 1_Y \circ f = f \) and \( f \circ 1_X = f \).

**Solution:** For all \( x \in X \) we have

\[
\begin{align*}
(1_Y \circ f)(x) &= 1_Y(f(x)) = f(x), \\
(f \circ 1_X)(x) &= f(1_X(x)) = f(x).
\end{align*}
\]

**Exercise 0.1.14** If \( f : X \to Y \) and \( g : Y \to Z \) are both one-to-one (or both onto) functions, then show that \( g \circ f \) is also one-to-one (or onto).
Solution: Assume that $f$ and $g$ are one-to-one. Let $x_1, x_2 \in X$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then we have $g(f(x_1)) = g(f(x_2))$. Now since $g$ is one-to-one, we must have $f(x_1) = f(x_2)$. Since $f$ is one-to-one, we deduce that $x_1 = x_2$. Therefore $g \circ f$ is one-to-one.

Assume that $f$ and $g$ are onto. We need to show that $g \circ f : X \rightarrow Z$ is onto. Let $z \in Z$. Since $g$ is onto (from $Y$ to $Z$), $\exists y \in Y$ such that $g(y) = z$.

Since $y \in Y$ and $f$ is onto (from $X$ to $Y$), $\exists x \in X$ such that $f(x) = y$. Now we have

$$(g \circ f)(x) = g(f(x)) = g(y) = z.$$ 

Hence $g \circ f$ is onto.

Exercise 0.1.15 (optional) Let $f : X \rightarrow Y$ be a function. Show that $f$ is invertible if and only if $f$ is both one-to-one and onto.

Solution: Assume that $f$ is invertible and its inverse is $f^{-1}$. We will show that $f$ is one-to-one and onto.

- $f$ is one-to-one: Because if $f(x) = f(\hat{x})$ for some $x, \hat{x} \in X$, then since $f^{-1}$ is a function we have $f^{-1}(f(x)) = f^{-1}(f(\hat{x}))$. That is $1_X(x) = 1_X(\hat{x})$, which implies $x = \hat{x}$.

- $f$ is onto: Because if $y \in Y$, then $f^{-1}(y) \in X$ and $f(f^{-1}(y)) = 1_Y(y) = y$.

Conversely assume that $f$ is one-to-one and onto. We will show that $f$ is invertible. Since $f$ is one-to-one and onto, for each $y$ in $Y$, there is a unique $x$ in $X$ such that $f(x) = y$. Define $f^{-1} : Y \rightarrow X$ by $f^{-1}(y) = x$. Then $f$ is a function and

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) \text{ and } (f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x.$$ 

So $f \circ f^{-1} = 1_X$ and $f^{-1} \circ f = 1_Y$. Thus $f^{-1}$ is the inverse of $f$. Hence $f$ is invertible.